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EQUILIBRIUM BIAS OF TECHNOLOGY

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Equilibrium Bias of Technology*

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Abstract

The study of the bias of new technologies is important both as part of the analysis of the nature of technology adoption and the direction of technological change, and to understand the distributional implications of new technologies. In this paper, I analyze the equilibrium bias of technology. I distinguish between the *relative bias* of technology, which concerns how the marginal product of a factor changes relative to that of another following the introduction of new technology, and the *absolute bias*, which looks only at the effect of new technology on the marginal product of a factor. The first part of the paper generalizes a number of existing results in the literature regarding the relative bias of technology. In particular, I show that when the menu of technological possibilities only allows for factor-augmenting technologies, the increase in the supply of a factor always induces technological change (or technology adoption) relatively biased towards that factor. This force can be strong enough to make the relative marginal product of a factor *increasing* in response to an increase in its supply, thus leading to an *upward-sloping* relative demand curve. However, I also show that the results about relative bias do not generalize when more general menus of technological possibilities are considered. In the second part of the paper, I show that there are much more general results about absolute bias. I prove that under fairly mild assumptions, an increase in the supply of a factor always induces changes in technology that are absolutely biased towards that factor, and these results hold both for small changes and large changes in supplies. Most importantly, I also determine the conditions under which the induced-technology response will be strong enough so that the price (marginal product) of a factor *increases* in response to an increase in its supply. These conditions correspond to a form of failure of joint concavity of the aggregate production function of the economy in factors and technology. This type of failure of joint concavity is quite possible in economies where equilibrium factor demands and technologies are decided by different agents.

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1 Introduction

Despite the generally-agreed importance of technological progress for economic growth and a large and influential literature on technological progress,[1] the determinants of the *direction* and *bias* of technological change are not well understood. An analysis of the direction and bias of technical change is important for a number of reasons. First, in most situations, technical change is not neutral: it benefits some factors of production, while directly or indirectly reducing the compensation of others. This possibility is illustrated both by the distributional impact of the major technologies introduced during the Industrial Revolution and the effects of technological change on the structure of wages during the past half century or so.[2] The bias of technological change determines its distributional implications (i.e., which groups are the winners and which will be the losers from technological progress) and thus the willingness of different groups to embrace new technologies. Second, an understanding of the determinants of innovation requires an analysis of the bias and direction of new technologies, for example, for evaluating whether lines of previous innovations or technologies will be exploited in the future and the potential compatibility between old and new technologies.[3] Finally, the bias of technology is important for understanding the macroeconomic implications of technological progress.

These and related questions have spurred a relatively large literature investigating various dimensions of the bias of technology. The pioneering study was Hicks' seminal book, *The Theory of Wages* (1932), which first discussed the issue of induced innovation.[4] The topic later attracted attention from the leading economists of the 1960s, notably Kennedy (1964), Samuelson (1965), Drandakis and Phelps (1965), Ahmad (1966), Nordhaus (1973), David (1975), and Binswanger and Ruttan (1978), who studied the link between factor prices and technical change. The focus of this literature was on the macroeconomic consequences of induced innovation and was shaped by a critical passage in Hicks's book where he argued:

[1] See, among others, Dasgupta and Stiglitz (1980), Reinganum (1981, 1985), Spence (1984), and Grossman and Shapiro (1987) in the industrial organization literature and Romer (1990), Segerstrom, Anant and Dinopoulos (1990), Grossman and Helpman (1991), Aghion and Howitt (1992), Stokey (1991, 1995), and Young (1993) in the economic growth literature.

[2] On the biases and distributional effects of the technologies introduced during the Industrial Revolution, see Mantoux (1961) or Mokyr (1990), and on recent developments, see footnote 6 below.

[3] See, for example, Farrell and Saloner (1985) and Katz and Shapiro (1985).

[4] There is an implicit reference to this issue in Marx, when he discusses how labor scarcity—the exhaustion of the reserve army of labor—may induce the capitalist to substitute machinery for labor (see Rosenberg, 1982), and also in Habakkuk's (1962) well-known contrast of faster technological progress in the United States than in Britain because of labor scarcity in the former country (see, in particular, p. 44).

> "A change in the relative prices of the factors of production is itself a spur to invention, and to invention of a particular kind—directed to economizing the use of a factor which has become relatively expensive." (pp. 124-5)

Although not explicitly stated, the implicit message in this sentence (and the way it was interpreted) was that factor prices were the crucial element shaping the bias and direction of technological progress (or technological adoption), and somehow as a factor becomes more abundant, thus less expensive, technical change should become *less biased* towards that factor.[5]

The topic of biased technological change received renewed interest over the past decade, as a result of a number of macro phenomena, particularly, the evidence that overall technological change over the past 60 years has been biased towards skilled workers (e.g., Autor, Katz and Krueger, 1998). This led a number of authors to formulate extensions of endogenous growth models (Acemoglu, 1998, 2002, 2003a,b, Acemoglu and Zilibotti, 2000, Kiley, 1999, Caselli and Coleman, 2004, Xu, 2001, Gancia, 2003, Thoenig and Verdier, 2003, Ragot, 2003, Duranton, 2004, and Jones, 2005), whereby technical change could be directed to one of multiple (typically two) sectors or factors.[6]

These models were descendents of the endogenous growth models of Romer, Grossman and Helpman, and Aghion and Howitt. As a result, they incorporated a number of specific features. These included a quasi-linear structure to obtain long-run growth, the constant elasticity of demand borrowed from the Dixit-Stiglitz-Spence model, different types of technologies that were of factor-augmenting type, and the market size effect, inherent in Romer's original article and present in the second generation of endogenous technical change models (see, e.g., Aghion and Howitt, 1998).

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[5] Nevertheless, parts of Hicks's reasoning did not go uncriticized. For example, Salter (1966) and Samuelson (1965) pointed out that firms should strive to economize on total costs not only on the factor that has become relatively more expensive, thus questioning Hicks's reasoning (see also Nordhaus, 1973). But the essential ideas encapsulated in the quote remained influential in the literature. I will clarify below that this quote is never correct for factor bias, but would be true for factor-augmenting changes as long as the relevant (local) elasticity of substitution is less than one.

Another implication of this quote relates to the effect of the price of a factor, say labor, on the overall amount of technology adoption or innovation, which is also an interesting area for study, but not part of the focus of this paper. See Acemoglu and Finkelstein (2005) for a theoretical and empirical investigation of this point.

[6] The focus of the first papers in this literature, Acemoglu (1998) and Kiley (1999), was to investigate when and why technology could be biased towards skilled workers. This was partly motivated by the evidence that in the 19th and early 20th centuries, new technologies were often replacing skilled workers, which contrasted to the later skill-biased nature of technological change (see James and Skinner, 1985, Goldin and Katz, 1995, on the earlier era, and Goldin and Katz, 1998 or Autor, Katz and Krueger, 1998, on the more recent trends). Later Acemoglu (2003b) and Jones (2005) used similar ideas to investigate whether there are any compelling reasons for technical change to be purely labor augmenting as required for the existence of a balanced growth path in standard growth models. Acemoglu (2003a), Xu (2001), Gancia (2003), and Thoenig and Verdier (2003) used versions of this framework to investigate the effect of international trade on the bias of technology.

This structure led to some very sharp results about the *relative equilibrium bias of technology*, which, in many ways, stood in contrast to the implicit message of Hicks's quote mentioned above. Using an endogenous growth model with two sectors and a constant elasticity of substitution between factors, Acemoglu (2002) showed that these models implied essentially the opposite conclusion to that of Hicks's quote. To describe these results, let relative bias, in a two-factor world, be the impact of new technology on the marginal product of a factor relative to that of the other.[7] The main result in this class of models is that when a factor becomes more abundant, technology becomes endogenously more (relatively) biased towards that factor.

The question that arises naturally is whether these results are an artifact of special assumptions imposed in this class of models. Understanding the source of existing results is not only important for deriving general theorems about equilibrium bias, but also because without such an understanding, the forces determining the nature of technology adoption and technological progress remain unclear. The purpose of this paper is to provide an in-depth analysis of equilibrium bias and provide general theorems in the most natural setting that allows an analysis of these questions.

To motivate the analysis, we may return to the results from Acemoglu (2002) mentioned above, and wonder whether those results hold in more general settings. In particular, we may start with the following conjecture:

Conjecture (Relative Endogenous Bias): When the supply of a factor Z increases, technology becomes *relatively* more biased towards factor Z.

The first theorem in this paper will show that this conjecture is correct in a world with two factors and where all technologies are of the factor-augmenting type, thus substantially generalizing existing results. Moreover, it will provide precise conditions for this relative bias to be *strong*, i.e., for the increase in the relative supply of a factor to increase its relative price once technology has adjusted to the change in factor supplies. The second result, however, is that once we depart from an environment in which all technologies are of the factor-augmenting kind, and in particular, once there is potentially a choice between technologies with different elasticities of substitution, this conjecture is no longer true. It is possible to construct relatively simple examples where it fails.

Despite this negative result, the rest of the paper presents a number of general results about equilibrium bias. In contrast to the focus in Acemoglu (1998, 2002) and the conjecture above,

[7] Equivalently, bias can be described as referring to cost-minimizing relative factor demands at a given factor price ratio. The two definitions of relative bias are equivalent for the purposes in this paper.

these results concern the "absolute", not relative, bias. A technology is said to be *absolutely biased* towards a factor if it increases its marginal product.[8] While understanding relative bias is essential for a certain class of questions (for example, those concerning inequality), an analysis of absolute bias is equally important, for example, for understanding the implications of technological change for the level of wages or the level of rewards to other factors.

One of the main results of the paper is the following theorem which is stated loosely and without the necessary assumptions here (which are explained subsequently):

Theorem (Weak Endogenous Bias): When the supply of a factor Z increases, technology becomes *absolutely* biased towards factor Z.

Stated differently, this theorem shows that under a set of relatively mild assumptions on the underlying environment, there is a strong result about equilibrium bias; technology will progressively favor factors that are becoming abundant. I will show that this theorem applies under two alternative sets of sufficient conditions. The first set of conditions requires the measure (vector) of technology to belong to a convex subset of $\mathbb{R}^K$ for some $K \geq 1$, which will lead to a local theorem (i.e., a result that applies in response to small changes in the supply of a factor Z). The second possibility is a global theorem. The conditions necessary for this version of the theorem can be best understood by using the tools of monotone comparative statics as developed by Topkis (1978, 1979, 1998), Milgrom and Roberts (1990a,b, 1994), Vives (1990), and Milgrom and Shannon (1994). In fact, the sufficient condition for the global theorem is a form of supermodularity (or increasing differences) between factor Z (or a set of factors, $Z_1, ..., Z_N$) and a vector θ denoting technology choices.

These results are not only interesting because of their generality, but also because they shed light on a variety of real-world phenomena. For example, they suggest why recent technical change may have increased the demand for skilled workers (since there has been a significant increase in the number of more educated workers), and why technological progress may have been biased towards unskilled workers in the past (since there was a large increase in the supply of unskilled workers in British cities during the 19th century, see, for example, Habakkuk 1962, or Williamson, 1990).

The above theorem is referred to as the "weak endogenous bias" theorem, because it only specifies the direction of the bias. Perhaps more important and certainly more surprising is the following theorem:

Theorem (Strong Endogenous Bias): Under sufficient non-convexity of the aggregate

[8] Thus the difference is that this marginal product is not compared to the marginal product of other factors.

production possibilities set of the economy, when the supply of a factor Z increases, technology becomes sufficiently biased towards factor Z so that the price of this factor increases.

In other words, with sufficient non-convexity of the production possibilities set, equilibrium bias can be strong enough that endogenous-technology demand curves for factors can be upward-sloping rather than downward-sloping as in the standard neoclassical theory. This result will be stated in Theorem 8 below, which not only shows the possibility of strong equilibrium (absolute) bias, but also makes the conditions under which this happens precise. In particular, this theorem shows that there will be strong (absolute) equilibrium bias if and only if the aggregate output of the economy (or a transformation of it) fails to be jointly concave in technology (say θ) and Z. In equilibrium output (profits) will be maximized in the choice of Z by firms, while the choice of technology θ by some other agents (a technology monopolist or research firm) will also maximize output (or some transform thereof). Nevertheless, these two conditions together do not guarantee that the equilibrium, say (Z^*, θ^*), is a maximum of the aggregate output (or its transform). Instead, (Z^*, θ^*) could be a *saddle point*, meaning that there exists a direction in the (Z, θ) plane in which aggregate output increases. Essentially, Theorem 8 will show that there will be strong equilibrium bias whenever this is the case, and strong bias will never exist when (Z^*, θ^*) is in fact a maximum. This implies that the equilibrium structure, where technology and factor demands are chosen by different agents in the economy is essential for this result, since otherwise, (Z^*, θ^*) would be a maximum, thus ruling out strong bias. Equally important, however, is the observation that once we have an economy in which factor demands and technology are chosen by different agents, such strong bias is quite easy to obtain, because there is nothing that guarantees that (Z^*, θ^*) should be a maximum in all directions.

In addition to the earlier work on induced innovation literature and the recent directed technical change literature that have already been discussed above, this paper is closely related to work on the LeChatelier principle. Recall that the LeChatelier principle concerns the demand for factors of a profit-maximizing firm, and states that long-run demand curves (which allow adjustment in all factors) are more elastic than short-run demand curves (which hold the employment level of other factors constant). In other words, in response to an increase in the price of a factor, the employment of this factor declines more in the short-run than in the long-run. This principle was first stated and proved by Samuelson (1947) for small changes in factor prices, but was known not to be true for large changes (see, for example, Samuelson, 1960, Roberts, 1999). It was later generalized by Milgrom and Roberts (1996) to a global

LeChatelier principle under the assumption that production functions are supermodular (see also Silberberg, 1974). The intuition underlying the LeChatelier principle is that the firm can adjust other factors to increase the marginal product of the factor whose price has increased. At some level, results about the endogenous bias of technology correspond to equilibrium versions of the LeChatelier principle. The main difference is that the focus here is the effect of changes in factor supplies on *equilibrium* outcomes, rather than the partial equilibrium/optimization focus of the LeChatelier principle. The above discussion illustrates that this equilibrium structure is responsible for the possibility of strong equilibrium bias (since a firm's demand curve for a factor can never be upward sloping, even in the long run, see, e.g., Mas-Colell, Winston and Green, 1995, Proposition 5.C.2). Equivalently, as discussed above, strong equilibrium bias requires technology and factor demands to be chosen by different agents.

The rest of the paper is organized as follows. In Section 2, I describe three alternative environments, with different market structures and assumptions on technology choice, and show that the determination of equilibrium bias in these three different economies boils down to the same problem, with the major difference that two of the economies allow for more natural non-convexities in the aggregate production possibilities set. Section 3 provides a significant generalization of existing relative bias results, but also shows why the conjecture regarding relative bias above is not correct unless we restrict the technology possibilities menu to only factor-augmenting technologies. Section 4 contains the main results on weak equilibrium (absolute) bias and presents a number of versions of the theorems, and also clarifies the limits of these theorems. Section 5 contains the results on strong equilibrium bias. Section 6 concludes, while Appendices A and B contain some additional technical material.

2 The Basic Environments

Consider a static economy consisting of a unique final good and two sets of factors of production, a total of $N+M$, $\mathbf{Z}=(Z_1,...,Z_N)$ and $\mathbf{L}=(L_1,...,L_M)$. Throughout, I assume that all agents' preferences are defined over the consumption of the final good. Moreover, all factors are supplied inelastically and denote their supplies by $\bar{\mathbf{Z}} \in \mathbb{R}_+^N$ and $\bar{\mathbf{L}} \in \mathbb{R}_+^M$. The reason for distinguishing between these two sets of factors is to carry out comparative static exercises varying the supply of factors $\mathbf{Z}$, while holding the supply of other factors, $\mathbf{L}$, constant. The economy consists of a continuum of firms (final good producers) denoted by the set $\mathcal{F}$, each with an identical production function. Without loss of any generality let us normalize the measure of $\mathcal{F}$, $|\mathcal{F}|$, to 1. The price of the final good is also normalized to 1.[9]

I will consider three different environments to highlight the importance of convexity of the aggregate production set. All three environments will lead to a similar structure for the determination of equilibrium bias. In particular, they will all generate the weak equilibrium bias under the set of conditions already discussed in the Introduction, but two of them can generate the strong equilibrium bias more naturally (see below for formal definitions).

The first, Economy D (for *decentralized*), is a completely decentralized economy in which technologies are chosen by firms themselves. In some ways, in this economy, technology choice can be interpreted as choice of just another set of factors. This economy also has some similarity to the models recently analyzed by Boldrin and Levine (2001, 2004) and Quah (2003), which emphasize the possibility of endogenous technological change without monopolistic competition. But from the point of view of this paper, the most important aspect of Economy D is that the whole discussion can be in terms of technology adoption, and we can work with a convex decentralized economy familiar from basic general equilibrium analysis.

The second, Economy C (for *centralized*), features a benevolent social planner choosing the technology. The third is, in many ways, the most standard environment, and features a monopolist choosing and selling technologies. This environment, Economy M (for *monopoly*), will lead to identical results to Economy C. A tradition dating back to Schumpeter (1934) and Arrow (1962), and more recently used by Romer (1990), Grossman and Helpman (1991), and Aghion and Howitt (1992), emphasizes both the non-rivalrous nature of new technologies and the monopoly power necessary to recoup the investments made for R&D, and these features are captured in Economy M in a simple manner.

2.1 Economy D—Decentralized Equilibrium

In the first environment, *Economy D,* all markets are competitive and technology is decided by each firm separately. In this case, each firm $i \in \mathcal{F}$ has access to a production function

$$Y^i = F(\mathbf{Z}^i, \mathbf{L}^i, \theta^i) \tag{1}$$

where $\mathbf{Z}^i \in \mathcal{Z} \subset \mathbb{R}_+^N, \mathbf{L}^i \in \mathcal{L} \subset \mathbb{R}_+^M$ and $\theta^i \in \Theta$ is the measure of technology. F is a real-valued production function, which, for simplicity, I take to be twice continuously differentiable in

[9] Since all agents' preferences are defined over the final good, ownership of firms is not important for the equilibrium allocations. In particular, firms will always maximize profits independent of their exact ownership structure. For this reason, I do not specify the ownership structure of firms in what follows.

$(\mathbf{Z}^i, \mathbf{L}^i)$.[10] For now I impose no structure on the set Θ, but for concreteness, one might think of $\Theta \subset \mathbb{R}^K$ for some $K \in \mathbb{N}$. For many instances of technology choice, Θ may consist of discrete elements (corresponding to separate technologies), so it may not be a convex set. For the global results, we will need that both Θ and $\mathcal{Z}$ are lattices according to some order.[11]

Each final good producer (firm) maximizes profits, i.e., it solves the problem:

$$\max_{\mathbf{Z}^i \in \mathcal{Z}, \mathbf{L}^i \in \mathcal{L}, \theta_i \in \Theta} \pi(\mathbf{Z}^i, \mathbf{L}^i, \theta^i) = F(\mathbf{Z}^i, \mathbf{L}^i, \theta^i) - \sum_{j=1}^{N} w_{Zj} Z_j^i - \sum_{j=1}^{M} w_{Lj} L_j^i, \tag{2}$$

where w_{Zj} is the price of factor Z_j for $j = 1, ..., N$, and w_{Lj} is the price of factor L_j for $j = 1, ..., M$, all taken as given by the firm. Similar to the notation for L and Z, I will use $\mathbf{w}_Z$ and $\mathbf{w}_L$ to denote the vector of factor prices. Since there is a total supply $\bar{Z}_j$ of factor Z_j and a total supply $\bar{L}_j$ of factor L_j, and both factors are supplied inelastically, market clearing requires

$$\int_{i \in \mathcal{F}} Z_j^i di \leq \bar{Z}_j \text{ for } j = 1, ..., N \text{ and } \int_{i \in \mathcal{F}} L_j^i di \leq \bar{L}_j \text{ for } j = 1, ..., M. \tag{3}$$

Definition 1 A competitive equilibrium in Economy D is a set of decisions $\{\mathbf{Z}^i, \mathbf{L}^i, \theta^i\}_{i \in \mathcal{F}}$ and factor prices $(\mathbf{w}_Z, \mathbf{w}_L)$ such that $\{\mathbf{Z}^i, \mathbf{L}^i, \theta^i\}_{i \in \mathcal{F}}$ solve (2) given prices $(\mathbf{w}_Z, \mathbf{w}_L)$ and (3) holds.

I refer to any θ^i that is part of the set of equilibrium allocations, $\{\mathbf{Z}^i, \mathbf{L}^i, \theta^i\}_{i \in \mathcal{F}}$, as "equilibrium technology".

Assumption 1 $\Theta \subset \mathbb{R}^K$ for some $K \geq 1$, $F(\mathbf{Z}^i, \mathbf{L}^i, \theta^i)$ is jointly strictly concave in $(\mathbf{Z}^i, \mathbf{L}^i, \theta^i)$ and increasing in $(\mathbf{Z}^i, \mathbf{L}^i)$, and $\mathcal{Z}$, $\mathcal{L}$ and Θ are convex.

Then by standard arguments we have:

Lemma 1 ***(Symmetry)*** Suppose Assumption 1 holds. Then in any competitive equilibrium, $(\mathbf{Z}^i, \mathbf{L}^i, \theta^i) = (\bar{\mathbf{Z}}, \bar{\mathbf{L}}, \theta)$ for all $i \in \mathcal{F}$.

[10] Whenever F is assumed to be differentiable in $(\mathbf{Z}^i, \mathbf{L}^i)$ $[(\mathbf{Z}^i, \mathbf{L}^i, \theta)]$, this means that it is differentiable over some open set containing $\mathcal{Z} \times \mathcal{L}$ $[\mathcal{Z} \times \mathcal{L} \times \Theta]$.

The differentiability assumptions are not necessary for the main results, and only facilitate the exposition by allowing a clear definition of marginal products and factor prices. Without differentiability, factor prices (marginal products) can take values in the set of generalized Clarke derivatives as defined in Clarke (1990).

[11] Since $\mathcal{Z}$ is a subset of $\mathbb{R}_+^N$, an easy way to guarantee that it is a lattice is to assume it to be a "box-constrained" region (or a cube) with a minimum and maximum value for each Z_j^i. Although the lattice structure can be restrictive under some circumstances, for example when there are budget-type relationships between subcomponents of the vector, it is not very restrictive in this context, since the fact that a firm is hiring more of one factor does not typically put constraints on its hiring more of others.

Proof. This lemma follows immediately by the strict concavity of $F(\mathbf{Z}^i, \mathbf{L}^i, \theta^i)$, which implies strict concavity of $\pi(\mathbf{Z}^i, \mathbf{L}^i, \theta^i)$. To obtain a contradiction, suppose that two firms, i and i', choose $(\mathbf{Z}^i, \mathbf{L}^i, \theta^i)$ and $(\mathbf{Z}^{i'}, \mathbf{L}^{i'}, \theta^{i'})$, such that $(\mathbf{Z}^i, \mathbf{L}^i, \theta^i) \neq (\mathbf{Z}^{i'}, \mathbf{L}^{i'}, \theta^{i'})$. This is only possible if $\pi(\mathbf{Z}^i, \mathbf{L}^i, \theta^i) = \pi(\mathbf{Z}^{i'}, \mathbf{L}^{i'}, \theta^{i'})$. Now consider the vector $(\mathbf{Z}, \mathbf{L}, \theta) = \lambda(\mathbf{Z}^i, \mathbf{L}^i, \theta^i) + (1-\lambda)(\mathbf{Z}^{i'}, \mathbf{L}^{i'}, \theta^{i'})$ for some $\lambda \in (0,1)$, which is feasible by the convexity of $\mathcal{L}$, $\mathcal{Z}$ and Θ. Strict concavity implies that $\pi(\mathbf{Z}, \mathbf{L}, \theta) > \lambda\pi(\mathbf{Z}^i, \mathbf{L}^i, \theta^i) + (1-\lambda)\pi(\mathbf{Z}^{i'}, \mathbf{L}^{i'}, \theta^{i'})$, hence $\pi(\mathbf{Z}, \mathbf{L}, \theta) > \pi(\mathbf{Z}^i, \mathbf{L}^i, \theta^i) = \pi(\mathbf{Z}^{i'}, \mathbf{L}^{i'}, \theta^{i'})$, delivering a contradiction. Therefore for all $i \in \mathcal{F}$, we have $(\mathbf{Z}^i, \mathbf{L}^i, \theta^i) = (\mathbf{Z}, \mathbf{L}, \theta)$. Since F is increasing in $(\mathbf{Z}^i, \mathbf{L}^i)$, market clearing, (3), and $|\mathcal{F}| = 1$ imply that $(\mathbf{Z}, \mathbf{L}) = (\bar{\mathbf{Z}}, \bar{\mathbf{L}})$, completing the proof. ■

Assumption 1 may be restrictive, however, because it rules out constant returns to scale in $(\mathbf{Z}^i, \mathbf{L}^i, \theta^i)$. Alternatively, we can modify this assumption to allow for constant returns to scale:[12]

Assumption 1' $\Theta \subset \mathbb{R}^K$ for some $K \geq 1$, $F(\mathbf{Z}^i, \mathbf{L}^i, \theta^i)$ is increasing in $(\mathbf{Z}^i, \mathbf{L}^i)$ and exhibits constant returns to scale in $(\mathbf{Z}^i, \mathbf{L}^i, \theta^i)$, and we have $(\bar{\mathbf{Z}}, \bar{\mathbf{L}}) \in \mathcal{Z} \times \mathcal{L}$.

Proposition 1 *(Welfare Theorem D)* Suppose Assumption 1 or Assumption 1' holds. Then any equilibrium technology θ is a solution to

$$\max_{\theta' \in \Theta} F(\bar{\mathbf{Z}}, \bar{\mathbf{L}}, \theta'), \tag{4}$$

and any solution to this problem is an equilibrium technology.

Proof. ($\Longrightarrow$) First suppose Assumption 1 holds. Suppose that $\{\mathbf{Z}^i, \mathbf{L}^i, \theta^i\}_{i \in \mathcal{F}}$ is a competitive equilibrium. By Lemma 1, $\{\mathbf{Z}^i, \mathbf{L}^i, \theta^i\}_{i \in \mathcal{F}}$ is such that $(\mathbf{Z}^i, \mathbf{L}^i, \theta^i) = (\bar{\mathbf{Z}}, \bar{\mathbf{L}}, \theta)$ for all $i \in \mathcal{F}$. Moreover, by the definition of a competitive equilibrium, there exist $\mathbf{w}_Z$ and $\mathbf{w}_L$ such that

$$(\bar{\mathbf{Z}}, \bar{\mathbf{L}}, \theta) \in \arg \max_{\mathbf{Z}^i \in \mathcal{Z}, \mathbf{L}^i \in \mathcal{L}, \theta_i \in \Theta} F(\mathbf{Z}^i, \mathbf{L}^i, \theta^i) - \sum_{j=1}^{N} w_{Zj} Z_j^i - \sum_{j=1}^{M} w_{Lj} L_j^i. \tag{5}$$

This implies that any equilibrium technology θ satisfies $\theta \in \arg\max_{\theta' \in \Theta} F(\bar{\mathbf{Z}}, \bar{\mathbf{L}}, \theta')$. Next, suppose that Assumption 1' holds. In that case, without loss of any generality, we can consider an equilibrium with only one (representative) firm active and employing $(\bar{\mathbf{Z}}, \bar{\mathbf{L}}) \in \mathcal{Z} \times \mathcal{L}$. Consequently, by the definition of a competitive equilibrium (5) holds. Thus the same conclusion follows.

[12] It is also possible to allow for mixtures of constant returns to scale and strict convexity, but this introduces additional notation, and since it is not essential for the focus here, I simplify the analysis by using either Assumption 1 or Assumption 1'.

($\Longleftarrow$) First suppose that Assumption 1 holds. Take $\theta \in \arg\max_{\theta' \in \Theta} F(\bar{\mathbf{Z}}, \bar{\mathbf{L}}, \theta')$. By the strict concavity of F, the first-order conditions of (5) are necessary and sufficient. Consider the factor price vectors $\mathbf{w}_Z$ and $\mathbf{w}_L$ such that $w_{Zj} = \partial F(\bar{\mathbf{Z}}, \bar{\mathbf{L}}, \theta)/\partial Z_j$ and $w_{Lj} = \partial F(\bar{\mathbf{Z}}, \bar{\mathbf{L}}, \theta)/\partial L_j$. The hypothesis (4) implies that at these factor price vectors, $(\mathbf{Z}^i, \mathbf{L}^i, \theta^i) = (\bar{\mathbf{Z}}, \bar{\mathbf{L}}, \theta)$ for all $i \in \mathcal{F}$ satisfies the first-order conditions of (5), so it is a competitive equilibrium, thus θ is an equilibrium technology. Next, suppose that Assumption 1' holds. Once again, we can consider an equilibrium with only one firm active employing $(\bar{\mathbf{Z}}, \bar{\mathbf{L}}) \in \mathcal{Z} \times \mathcal{L}$, so any $\theta \in \arg\max_{\theta' \in \Theta} F(\bar{\mathbf{Z}}, \bar{\mathbf{L}}, \theta')$ is an equilibrium technology, completing the proof. ■

Proposition 1 is useful since it enables us to focus on a simple maximization problem rather than an equilibrium problem. An important implication of this proposition is also that the equilibrium corresponds to a maximum of F in the entire vector $(\mathbf{Z}^i, \mathbf{L}^i, \theta^i)$. It is also straightforward to see that equilibrium factor prices in this economy are equal to the marginal products of the F function, and are given by $w_{Zj} = \partial F(\bar{\mathbf{Z}}, \bar{\mathbf{L}}, \theta)/\partial Z_j$ and $w_{Lj} = \partial F(\bar{\mathbf{Z}}, \bar{\mathbf{L}}, \theta)/\partial L_j$ where θ is the equilibrium technology choice.

We next derive a similar maximization problem for Economies C and M, which relax the strong (joint) convexity assumptions inherent in Economy D, and show that a similar equilibrium characterization can be obtained for these economies, but without the implication that the equilibrium corresponds to a maximum of F in the entire vector $(\mathbf{Z}^i, \mathbf{L}^i, \theta^i)$.

2.2 Economy C—Centralized Equilibrium

In this economy, there is still a unique final good and each firm has access to the production function

$$Y^i = G(\mathbf{Z}^i, \mathbf{L}^i, \theta^i). \tag{6}$$

In particular, we again have $\mathbf{Z}^i \in \mathcal{Z} \subset \mathbb{R}_+^N$, $\mathbf{L}^i \in \mathcal{L} \subset \mathbb{R}_+^M$ and $\theta^i \in \Theta$ is the measure of technology, and G is again a real-valued production function that is twice continuously differentiable in $(\mathbf{Z}^i, \mathbf{L}^i)$.

Each firm has free access to the technology θ provided by the centralized (socially-run) research firm. This research firm can create any technology θ at cost $C(\theta)$ from the available technology menu Θ. Once created, this technology is non-excludable and available to any firm (as well as non-rival, see Arrow, 1962, Romer, 1990). In addition, to further simplify the analysis, I assume that the research firm can only choose one technology, which might be, for example, because of the necessity of standardization across firms.[13]

[13] In general, a social planner may want to create two different technologies, say θ_1 and θ_2, and provide one

All factor markets are again competitive. Consequently, given the technology offer of θ of the research firm, the maximization problem of each final good producer is

$$\max_{\mathbf{Z}^i \in \mathcal{Z}, \mathbf{L}^i \in \mathcal{L}} \pi(\mathbf{Z}^i, \mathbf{L}^i, \theta) = G(\mathbf{Z}^i, \mathbf{L}^i, \theta) - \sum_{j=1}^{N} w_{Zj} Z_j^i - \sum_{j=1}^{M} w_{Lj} L_j^i. \tag{7}$$

Notice the important difference in this maximization problem relative to that in Economy D: firms are only maximizing with respect to $(\mathbf{Z}^i, \mathbf{L}^i)$, not with respect to θ^i, which will be determined by the research firm.

The objective of the research firm is to maximize total surplus, or total output. Since $\theta^i = \theta$ for all $i \in \mathcal{F}$, this is equivalent to

$$\max_{\theta^i \in \Theta} \Pi(\theta) = \int_0^1 G(\mathbf{Z}^i, \mathbf{L}^i, \theta) di - C(\theta). \tag{8}$$

This leads to a natural definition of equilibrium:

Definition 2 An equilibrium in Economy C is a set of firm decisions $\{\mathbf{Z}^i, \mathbf{L}^i\}_{i \in \mathcal{F}}$, technology choice θ and factor prices $(\mathbf{w}_Z, \mathbf{w}_L)$ such that $\{\mathbf{Z}^i, \mathbf{L}^i\}_{i \in \mathcal{F}}$ solve (7) given $(\mathbf{w}_Z, \mathbf{w}_L)$ and θ, (3) holds, and the technology choice for the research firm, θ, maximizes (8).

We now impose weaker versions of Assumptions 1 and 1' on G:

Assumption 2 $G(\mathbf{Z}^i, \mathbf{L}^i, \theta^i)$ is jointly strictly concave and increasing in $(\mathbf{Z}^i, \mathbf{L}^i)$ and $\mathcal{Z}$ and $\mathcal{L}$ are convex.

Assumption 2' $G(\mathbf{Z}^i, \mathbf{L}^i, \theta^i)$ is increasing and exhibits constant returns to scale in $(\mathbf{Z}^i, \mathbf{L}^i)$, and we have $(\bar{\mathbf{Z}}, \bar{\mathbf{L}}) \in \mathcal{Z} \times \mathcal{L}$.

The important difference between Assumptions 1 and 1' versus Assumptions 2 and 2' is that with the latter, $G(\mathbf{Z}^i, \mathbf{L}^i, \theta^i)$ does not need to be jointly concave in $(\mathbf{Z}^i, \theta)$, which will play an important role in the results below (nor does Θ need to be a subset of $\mathbb{R}^K$).

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technology to a subset of firms and the other to the rest. This strategy may be optimal if $C(\theta)$ were sufficiently small (so that duplication costs are not too large).

In the environment outlined here, this option will not typically work because of non-excludability. In particular, all firms would want to use the technology that is superior. Nevertheless, there can be some situations in which the research firm may prefer to create two distinct technologies, θ_1 and θ_2, from the menu. For this, it needs to be the case that *first,* neither of the two technologies is superior to the other (i.e., which one leads to higher output depends on factor proportions); *second,* F has to be jointly non-concave in θ and $(\mathbf{Z}, \mathbf{L})$, so that some firms may choose θ_1 and the corresponding factor demand, while others choose θ_2 and other levels of factor demands, and all firms make the same level of profits; and *third,* $C(\theta)$ should be low enough that the costs of creating two different technologies, θ_1 and θ_2, is not prohibitive. Such situations are relatively rare and are not central to the focus here, so rather than deriving the conditions on $C(\theta)$ and the production function G to rule out this possibility, I simply assume that choosing two separate technologies from the menu is not possible.

Proposition 2 *(Equilibrium Theorem C)* Suppose Assumption 2 or Assumption 2' holds. Then any equilibrium technology is a solution to

$$\max_{\theta\in\Theta} G(\bar{\mathbf{Z}},\bar{\mathbf{L}},\theta) - C(\theta) \tag{9}$$

and any solution to this problem is an equilibrium technology.

Proof. The proof is similar to that of Proposition 1, and follows again by noting that under Assumption 2, the equilibrium will be symmetric, so $(\mathbf{Z}^i,\mathbf{L}^i,\theta) = (\mathbf{Z},\mathbf{L},\theta)$. In addition, because G is increasing in $(\mathbf{Z}^i,\mathbf{L}^i)$, market clearing, (3), yields that $(\mathbf{Z},\mathbf{L}) = (\bar{\mathbf{Z}},\bar{\mathbf{L}})$, which implies that (8) is identical to (9). When Assumption 2' holds, there are constant returns to scale in $(\mathbf{Z},\mathbf{L})$, and $(\bar{\mathbf{Z}},\bar{\mathbf{L}}) \in \mathcal{Z}\times\mathcal{L}$, so we can once again work with a single firm employing $(\bar{\mathbf{Z}},\bar{\mathbf{L}})$, and the conclusion follows. ■

Defining $F(\bar{\mathbf{Z}},\bar{\mathbf{L}},\theta) = G(\bar{\mathbf{Z}},\bar{\mathbf{L}},\theta) - C(\theta)$, we obtain that technology choice in Economy C can be characterized as maximizing some function $F(\bar{\mathbf{Z}},\bar{\mathbf{L}},\theta)$ with respect to $\theta\in\Theta$ as in Economy D. However, I refer to this as an "equilibrium theorem" not as a welfare theorem as for Proposition 1, since despite the fact that the objective of the research firm is to maximize social surplus, the equilibrium may not be the social optimum. This results from the fact that once created, technologies are non-excludable, so all firms use it, whereas the social planner may have preferred to exclude some firms to enable remaining firms to hire more of the factors of production (recall footnote 13). The equilibrium structure, as captured by Definition 2, does not allow for this possibility given the non-excludable nature of the technology.

For our purposes, the more important difference is that while in Economy D $F(\bar{\mathbf{Z}},\bar{\mathbf{L}},\theta)$ is by assumption jointly concave in $(\mathbf{Z},\theta)$, the same is not true in Economy C. In particular, in this latter economy, $F(\bar{\mathbf{Z}},\bar{\mathbf{L}},\theta)$ does not need to be concave in $(\mathbf{Z},\theta)$ (nor is it necessarily globally concave in θ).

It is also useful to note that equilibrium factor prices are now given by $w_{Zj} = \partial G(\bar{\mathbf{Z}},\bar{\mathbf{L}},\theta)/\partial Z_j$ and $w_{Lj} = \partial G(\bar{\mathbf{Z}},\bar{\mathbf{L}},\theta)/\partial L_j$, but since $F(\bar{\mathbf{Z}},\bar{\mathbf{L}},\theta) = G(\bar{\mathbf{Z}},\bar{\mathbf{L}},\theta) - C(\theta)$, this is equivalent to $w_{Zj} = \partial F(\bar{\mathbf{Z}},\bar{\mathbf{L}},\theta)/\partial Z_j$ and $w_{Lj} = \partial F(\bar{\mathbf{Z}},\bar{\mathbf{L}},\theta)/\partial L_j$ as in Economy D.

2.3 Economy M—Monopoly Equilibrium

Now I briefly discuss an economy that is similar to Economy C, but features a monopolist supplying technologies to final good producer firms. I take the simplest structure to deliver results similar to Propositions 1 and 2, while Appendix A analyzes exactly the setup of Economy C with a monopolist provider of technologies.

In the environment here, there is still a unique final good and each firm has access to the production function

$$Y^i = \alpha^{-\alpha}(1-\alpha)^{-1}\left[G(\mathbf{Z}^i, \mathbf{L}^i, \theta^i)\right]^\alpha q\left(\theta^i\right)^{1-\alpha} \tag{10}$$

which is similar to (6), except that $G(\mathbf{Z}^i, \mathbf{L}^i, \theta^i)$ is now a subcomponent of the production function, which depends on θ^i, the technology being used by the firm. This subcomponent needs to be combined with an intermediate good embodying technology θ^i, denoted by $q\left(\theta^i\right)$— conditioned on θ^i to emphasize that it embodies technology θ^i. This intermediate good is supplied by the monopolist. The term $\alpha^{-\alpha}(1-\alpha)^{-1}$ in the front is a convenient normalization. This structure is a generalization of the setup common in equilibrium models of endogenous technology (e.g., Romer, 1990, Grossman and Helpman, 1991, or Aghion and Howitt, 1992, 1998). As before, I assume that $\mathbf{Z}^i \in \mathcal{Z} \subset \mathbb{R}_+^N, \mathbf{L}^i \in \mathcal{L} \subset \mathbb{R}_+^M$ and G is a real-valued production function that is twice continuously differentiable in $(\mathbf{Z}^i, \mathbf{L}^i)$.

The technology monopolist can create technology θ at cost $C(\theta)$ from the technology menu, and again I assume that it can only choose one technology. Once created, the technology monopolist can produce as many units of the intermediate good of type θ (that is, of the intermediate goods embodying technology θ) at per unit cost normalized to $1-\alpha$ unit of the final good (this is also a convenient normalization, without any substantive implications). It can then set a (linear) price per unit of the intermediate good of type θ, denoted by χ.

All factor markets are again competitive. Consequently, each firm takes the type of available technology, θ, and the price of the intermediate good embodying this technology, χ, as given and maximizes

$$\max_{\substack{\mathbf{Z}^i \in \mathcal{Z}, \mathbf{L}^i \in \mathcal{L}, \\ q(\theta) \geq 0}} \pi(\mathbf{Z}^i, \mathbf{L}^i, q(\theta) \mid \theta, \chi) = \alpha^{-\alpha}(1-\alpha)^{-1}\left[G(\mathbf{Z}^i, \mathbf{L}^i, \theta)\right]^\alpha q(\theta)^{1-\alpha} - \sum_{j=1}^N w_{Zj} Z_j^i - \sum_{j=1}^M w_{Lj} L_j^i - \chi q(\theta), \tag{11}$$

which gives the following simple inverse demand for intermediates of type θ as a function of its price, χ, and the factor employment levels of the firm as

$$q^i\left(\theta, \chi, \mathbf{Z}^i, \mathbf{L}^i\right) = \alpha^{-1} G(\mathbf{Z}^i, \mathbf{L}^i, \theta) \chi^{-1/\alpha}. \tag{12}$$

The problem of the monopolist is to maximize its profits (which are equal to price minus marginal cost of production times total sales of the intermediates, minus the cost of creating the technology). Thus the problem of the monopolist is:

$$\max_{\theta, \chi, [q^i(\theta, \chi, \mathbf{Z}^i, \mathbf{L}^i)]_{i \in \mathcal{F}}} \Pi = (\chi - (1-\alpha)) \int_{i \in \mathcal{F}} q^i\left(\theta, \chi, \mathbf{Z}^i, \mathbf{L}^i\right) di - C(\theta) \tag{13}$$

subject to (12). Therefore, an equilibrium in this economy can be defined as:

Definition 3 An equilibrium in Economy M is a set of firm decisions $\{\mathbf{Z}^i, \mathbf{L}^i, q^i(\theta, \chi, \mathbf{Z}^i, \mathbf{L}^i)\}_{i\in\mathcal{F}}$, technology choice θ, and factor prices $(\mathbf{w}_Z, \mathbf{w}_L, \chi)$ such that $\{\mathbf{Z}^i, \mathbf{L}^i, q^i(\theta, \chi, \mathbf{Z}^i, \mathbf{L}^i)\}_{i\in\mathcal{F}}$ solve (11) given $(\mathbf{w}_Z, \mathbf{w}_L, \chi)$ and technology θ, (3) holds, and the technology choice and pricing decision for the monopolist, (θ, χ), maximize (13) subject to (12).

Once again the important distinction between this definition and Definition 1 is that now factor demands and technology are being decided by different agents (the former by the final good producers, the latter by the technology monopolist).

The equilibrium in this economy is straightforward to characterize because (12) defines a constant elasticity demand curve, so the optimal price of the monopolist that maximizes (13) is simply the standard monopoly markup, i.e., $1/(1-\alpha)$ times the marginal cost of production of the intermediate, $1-\alpha$. This leads to an equilibrium monopoly price of $\chi = 1$. Moreover, I continue to impose Assumption 2 or 2' (which now apply to G, the subcomponent of the production function (10)). Under these assumptions, the equilibrium will again be symmetric, so $q^i(\theta, \chi) = \alpha^{-1}G(\bar{\mathbf{Z}}, \bar{\mathbf{L}}, \theta)\chi^{-1/\alpha}$ for all $i \in \mathcal{F}$, and given the monopoly price $\chi = 1$, we have $q^i(\theta) = q^i(\theta, \chi = 1, \bar{\mathbf{Z}}, \bar{\mathbf{L}}) = \alpha^{-1}G(\bar{\mathbf{Z}}, \bar{\mathbf{L}}, \theta)$ for all $i \in \mathcal{F}$. The profits and the maximization problem of the monopolist can then be expressed as

$$\max_{\theta\in\Theta} \Pi(\theta) = G(\bar{\mathbf{Z}}, \bar{\mathbf{L}}, \theta) - C(\theta). \tag{14}$$

Thus we have established (proof in the text):

Proposition 3 ***(Equilibrium Theorem M)*** Suppose Assumption 2 or Assumption 2' holds. Then an equilibrium in Economy M is a solution to

$$\max_{\theta\in\Theta} G(\bar{\mathbf{Z}}, \bar{\mathbf{L}}, \theta) - C(\theta)$$

and any solution to this problem is an equilibrium.

Relative to Economies D and C, the presence of the monopoly markup implies the presence of greater distortions in this economy.[14] More important for our purposes here, however, is that again defining $F(\bar{\mathbf{Z}}, \bar{\mathbf{L}}, \theta) = G(\bar{\mathbf{Z}}, \bar{\mathbf{L}}, \theta) - C(\theta)$, equilibrium technology in Economy M is a solution to a problem identical to that in Economy C, and quite similar to the one in Economy

[14]For example, it can be verified that taking the behavior of the final good producers as given, the socially optimal allocation in this case would maximize $(1-\alpha)^{-1/\alpha}[G(\bar{\mathbf{Z}}, \bar{\mathbf{L}}, \theta^i)] - C(\theta)$ rather than $[G(\bar{\mathbf{Z}}, \bar{\mathbf{L}}, \theta^i)] - C(\theta)$.

D. As in Economy C, $F(\bar{\mathbf{Z}}, \bar{\mathbf{L}}, \theta)$ need not be globally concave in θ nor even locally concave in $(\mathbf{Z}, \theta)$ in the neighborhood of the equilibrium.

This result therefore shows that for the analysis of equilibrium bias, it is not important whether technology choices are at the firm level or at the centralized level (resulting from some R&D or other research process), and also whether they are made to maximize social surplus or monopoly profits. But we will see that whether $F(\bar{\mathbf{Z}}, \bar{\mathbf{L}}, \theta)$ is jointly concave in $(\mathbf{Z}, \theta)$ will play an important role in the results.

Finally, it can be verified that in this economy equilibrium factor prices are given by $w_{Zj} = (1-\alpha)^{-1} \partial G(\bar{\mathbf{Z}}, \bar{\mathbf{L}}, \theta)/\partial Z_j$ and $w_{Lj} = (1-\alpha)^{-1} \partial G(\bar{\mathbf{Z}}, \bar{\mathbf{L}}, \theta)/\partial L_j$, which are proportional to the derivatives of the F function defined as $F(\bar{\mathbf{Z}}, \bar{\mathbf{L}}, \theta) = G(\bar{\mathbf{Z}}, \bar{\mathbf{L}}, \theta) - C(\theta)$. So to facilitate comparison with Economies D and C, with a slight abuse of terminology I will refer to the derivatives of the F function as the "equilibrium factor prices" even in Economy M.

3 Relative Equilibrium Bias

The previous section established that in three different environments, with different market structures and conceptions of technology choice, the characterization of equilibrium technology boils down to an identical maximization problem—the maximization of some function $F(\bar{\mathbf{Z}}, \bar{\mathbf{L}}, \theta)$ where $\bar{\mathbf{Z}}$ and $\bar{\mathbf{L}}$ are the factor supplies in the economy. In this and the next two sections, I make use of this characterization to derive a number of results about equilibrium bias of technology choice.

This section analyzes relative equilibrium bias, and for that reason, throughout I focus on a more specialized economy with only two factors, L and Z (i.e., $M = 1$ and $N = 1$), and $\theta \in \Theta \subset \mathbb{R}^K$ for some $K \geq 1$, so that $F : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^K \to \mathbb{R}_+$. Moreover, suppose that $Z \in \mathcal{Z} \subset \mathbb{R}_+$, $L \in \mathcal{L} \subset \mathbb{R}_+$, and that Θ is a convex compact subset of $\mathbb{R}^K$ with the jth component denoted by θ_j. Finally, I assume that F is twice continuously differentiable in (Z, L, θ).

Recall that, in a two-factor economy, *relative equilibrium bias* is defined as the effect of technology on the marginal product (price) of a factor relative to the marginal product (price) of the other factor. Denote the marginal product (or price) of the two factors by

$$w_Z(Z, L, \theta) = \frac{\partial F(Z, L, \theta)}{\partial Z} \text{ and } w_L(Z, L, \theta) = \frac{\partial F(Z, L, \theta)}{\partial L},$$

when employment levels (factor proportions) are given by (Z, L) and the technology is θ.[15]

[15]Recall that $F(\bar{\mathbf{L}}, \bar{\mathbf{Z}}, \theta)$ either corresponds to the production function of the firms (Economy D) or we have

From the twice differentiability of F, these marginal products are also differentiable functions of Z and L. Then we have the following definitions:[16]

Definition 4 An increase in technology θ_j for some $j = 1, ..., K$ is *relatively biased* towards factor Z at $(\bar{Z}, \bar{L}, \theta) \in \mathcal{Z} \times \mathcal{L} \times \Theta$ if

$$\frac{\partial w_Z(\bar{Z}, \bar{L}, \theta) / w_L(\bar{Z}, \bar{L}, \theta)}{\partial \theta_j} \geq 0.$$

This definition simply expresses what it means for a technology to be relatively biased towards a factor (similarly a decrease in θ_j is relatively biased towards factor Z, if the derivative in Definition 4 is non-positive). From this definition, it is clear that (weak) relative equilibrium bias should correspond to a change in technology θ in a direction biased towards Z *in response to an increase* in $\bar{Z}$ (or $\bar{Z}/\bar{L}$); this is stated in the next definition.[17]

Definition 5 Denote the equilibrium technology at factor supplies $(\bar{Z}, \bar{L}) \in \mathcal{Z} \times \mathcal{L}$ by $\theta(\bar{Z}, \bar{L})$, and assume that $\partial\theta_j(\bar{Z}, \bar{L}) / \partial Z$ exists at $(\bar{Z}, \bar{L})$ for all for all $j = 1, ..., K$. Then there is *relative equilibrium bias* at $(\bar{Z}, \bar{L}, \theta(\bar{Z}, \bar{L}))$ if

$$\sum_{j=1}^{K} \frac{\partial w_Z(\bar{Z}, \bar{L}, \theta(\bar{Z}, \bar{L})) / w_L(\bar{Z}, \bar{L}, \theta(\bar{Z}, \bar{L}))}{\partial \theta_j} \frac{\partial \theta_j(\bar{Z}, \bar{L})}{\partial Z} \geq 0. \quad (15)$$

Notice that the definition of relative equilibrium bias requires the (overall) change in technology in response to an increase in $\bar{Z}/\bar{L}$ to be biased towards Z at the point $(\bar{Z}, \bar{L}) \in \mathcal{Z} \times \mathcal{L}$ for which $\partial\theta_j(\bar{Z}, \bar{L}) / \partial Z$ exists for all j. The statement is not qualified with "towards Z" since relative equilibrium bias is also equivalent to a decline in Z (or Z/L) inducing a change in technology relatively biased against Z. Finally, the requirement that $\partial\theta_j(\bar{Z}, \bar{L}) / \partial Z$ exists for all j used in this definition will be further discussed below (in particular, see the discussion after Theorem 3 in the next section).

$F(\mathbf{\bar{L}}, \mathbf{\bar{Z}}, \theta) = G(\mathbf{\bar{L}}, \mathbf{\bar{Z}}, \theta) - C(\theta)$, where $G(\mathbf{\bar{L}}, \mathbf{\bar{Z}}, \theta)$ is the production function of the firms (Economy C) or a subcomponent of the production function (Economy M). In both cases, the derivatives of F with respect to $\mathbf{Z}$ and $\mathbf{L}$ define the marginal products of these factors. With a slight abuse of terminology, I will refer to $F(\mathbf{\bar{L}}, \mathbf{\bar{Z}}, \theta)$ as "the production function".

[16] For this section, all definitions are "*local*" in the sense that, I will only look at the effect of small changes in factor supplies. This is why they are expressed in terms of derivatives. I do not add this qualification to simplify terminology.

[17] Throughout this section, I focus on changes in the supply of factor Z, which is also equivalent to a change in relative supplies Z/L (with L kept constant). Consequently, I use ∂Z and $\partial(Z/L)$ interchangeablely. Moreover, I denote the change in equilibrium technology by $\partial\theta_j(\bar{Z}, \bar{L}) / \partial Z$ (or $\partial\theta_j(\bar{Z}, \bar{L}) / \partial(Z/L)$) rather than $d\theta_j(\bar{Z}, \bar{L}) / dZ$ (or $d\theta_j(\bar{Z}, \bar{L}) / d(Z/L)$) since θ is not generally only a function of Z or Z/L. I reserve the notation $d(w_Z/w_L)/dZ$ to denote the total change in relative (or absolute) wages, which includes the technological adjustment, and contrast this with the partial change $\partial(w_Z/w_L)/\partial Z$, which holds technology constant (see, for example, (16)).

The next definition introduces the more stringent concept of strong relative bias, which requires that in response to an increase in $\bar{Z}$ (or $\bar{Z}/\bar{L}$), technology changes so much that the overall effect (after the induced change in technology) is to increase the relative price of factor Z.

Definition 6 Denote the equilibrium technology at factor supplies $(\bar{Z}, \bar{L}) \in \mathcal{Z} \times \mathcal{L}$ by $\theta(\bar{Z}, \bar{L})$, and assume that $\partial\theta_j(\bar{Z}, \bar{L})/\partial(Z/L)$ exists at $(\bar{Z}, \bar{L})$ for all $j = 1, ..., K$. Then there is *strong relative equilibrium bias* at $(\bar{Z}, \bar{L}, \theta(\bar{Z}, \bar{L}))$ if

$$\begin{aligned} \frac{dw_Z(\bar{Z}, \bar{L}, \theta(\bar{Z}, \bar{L}))/w_L(\bar{Z}, \bar{L}, \theta(\bar{Z}, \bar{L}))}{dZ} &= \frac{\partial w_Z(\bar{Z}, \bar{L}, \theta(\bar{Z}, \bar{L}))/w_L(\bar{Z}, \bar{L}, \theta(\bar{Z}, \bar{L}))}{\partial Z} + \\ &\sum_{j=1}^{K} \frac{\partial w_Z(\bar{Z}, \bar{L}, \theta(\bar{Z}, \bar{L}))/w_L(\bar{Z}, \bar{L}, \theta(\bar{Z}, \bar{L}))}{\partial\theta_j} \frac{\partial\theta_j(\bar{Z}, \bar{L})}{\partial Z} > 0. \end{aligned} \tag{16}$$

By comparing the latter two definitions, it is clear that there will be strong relative equilibrium bias if the sum of the expressions in (15) over $j = 1, ..., K$ is large enough to dominate the direct (negative) effect of the increase in relative supplies on relative wages (which is the first term in (16)).

The main result in this section is that the conjecture about relative equilibrium bias applies in a world with only factor-augmenting technologies, but not more generally. Before deriving these results, it is useful to clarify the notions introduced so far using an example, which captures the main findings in Acemoglu (1998, 2002), but in the context of Economy C or M studied above rather than in the models of the original papers. In particular, the next example considers an environment equivalent to Economy C or M above, with constant returns to scale in L and Z. This example will also clarify one possible meaning of the Hicks's quote discussed above (see in particular footnote 20).

Example 1 ***(Relative Equilibrium Bias)*** Suppose that

$$G(Z, L, \theta) = \left[\gamma(A_Z Z)^{\frac{\sigma-1}{\sigma}} + (1-\gamma)(A_L L)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}, \tag{17}$$

where $\theta = (A_Z, A_L)$. In particular, A_Z and A_L are two separate factor-augmenting technology terms, $\gamma \in (0, 1)$ is a distribution parameter which determines how important the two factors are, and $\sigma \in (0, \infty)$ is the elasticity of substitution between the two factors. When $\sigma = \infty$, the two factors are perfect substitutes, and the production function is linear. When $\sigma = 1$, the production function is Cobb-Douglas, and when $\sigma = 0$, there is no substitution between the two factors, and the production function is Leontieff. Since there are two technology terms, I take $\theta = (A_Z, A_L) \in \Theta = \mathbb{R}^2_+$.

Suppose that factor supplies are given by $(\bar{Z}, \bar{L})$. Then the relative marginal product of the two factors is:

$$\frac{w_Z}{w_L} = \frac{\gamma}{1-\gamma}\left(\frac{A_Z}{A_L}\right)^{\frac{\sigma-1}{\sigma}}\left(\frac{\bar{Z}}{\bar{L}}\right)^{-\frac{1}{\sigma}}. \tag{18}$$

The relative marginal product of Z is decreasing in the relative abundance of Z, $\bar{Z}/\bar{L}$. This is the usual substitution effect, leading to a downward-sloping relative demand curve. This expression also makes it clear that the measure of relative bias towards Z will correspond to $\bar{\theta} = (A_Z/A_L)^{(\sigma-1)/\sigma}$,[18] since higher levels of $\bar{\theta}$ increase the marginal product of Z relative to labor for all values of σ (recall Definition 4). To derive the results similar to those in Acemoglu (1998, 2002) in the context of Economy C or M, suppose that the costs of producing new technologies are $\eta_Z A_Z^{1+\delta}$ and $\eta_L A_L^{1+\delta}$, where $\delta > 0$. Despite the fact that $\delta > 0$ introduces diminishing returns in the choice of technology, the production possibilities set of this economy is non-convex, since there is choice both over the factors of production, Z and L, and the technologies, A_Z and A_L (so that the function (17) exhibits increasing returns in L, Z, A_Z and A_L). From Proposition 2 or 3, the problem of choosing equilibrium technology is the following strictly concave maximization problem:

$$\max_{A_Z, A_L}\left[\gamma\left(A_Z\bar{Z}\right)^{\frac{\sigma-1}{\sigma}} + (1-\gamma)\left(A_L\bar{L}\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} - \eta_Z A_Z^{1+\delta} - \eta_L A_L^{1+\delta}.$$

Taking the ratio of the first-order necessary and sufficient conditions with respect to A_Z and A_L, and denoting the equilibrium values by *'s, the solution to this problem yields

$$\frac{A_Z^*}{A_L^*} = \left(\frac{\eta_Z}{\eta_L}\right)^{-\frac{\sigma}{1+\sigma\delta}}\left(\frac{\gamma}{1-\gamma}\right)^{\frac{\sigma}{1+\sigma\delta}}\left(\frac{\bar{Z}}{\bar{L}}\right)^{\frac{\sigma-1}{1+\sigma\delta}}. \tag{19}$$

This equation can also be expressed in an alternative form, both useful for the discussion here and for Theorem 1 below:

$$\frac{\partial \ln\left(A_Z^*/A_L^*\right)}{\partial \ln\left(Z/L\right)} = \frac{\sigma-1}{1+\sigma\delta}. \tag{20}$$

Now using equation (18) and Definition 4, we can express the condition for weak relative

[18] Alternatively, we could define $A_Z^{(\sigma-1)/\sigma}$ and $A_L^{-(\sigma-1)/\sigma}$ as two separate technology terms, both relatively biased towards Z, but clearly focusing on $\bar{\theta}$ is more economical.

It is important that the bias towards factor Z is $\bar{\theta} = (A_Z/A_L)^{(\sigma-1)/\sigma}$, *not* A_Z/A_L, as is sometimes confusingly and incorrectly stated in the applied literature. A_Z/A_L is the ratio of Z-augmenting to L-augmenting technology. When $\sigma > 1$, an increase in A_Z/A_L increases the relative marginal product of Z, while when $\sigma < 1$, an increase in A_Z/A_L reduces the relative marginal product of Z.

equilibrium bias as:[19]

$$\Delta\left(w_Z/w_L\right) \equiv \frac{\partial \ln\left(w_Z\left(\bar{Z},\bar{L},A_Z^*,A_L^*\right)/w_L\left(\bar{Z},\bar{L},A_Z^*,A_L^*\right)\right)}{\partial \ln\left(A_Z/A_L\right)}\frac{\partial \ln\left(A_Z^*/A_L^*\right)}{\partial \ln\left(Z/L\right)} \geq 0. \tag{21}$$

Using (18) and (20),

$$\Delta\left(w_Z/w_L\right) = \frac{\sigma-1}{\sigma}\times\frac{\sigma-1}{1+\sigma\delta} = \frac{(\sigma-1)^2}{(1+\sigma\delta)\,\sigma} \geq 0,$$

which is always nonnegative, thus establishing that there is always *weak relative equilibrium bias* (as claimed by the conjecture in the Introduction). Alternatively, the same result follows by looking directly at the measure of relative biased towards Z introduced above, $\bar{\theta} = \left(A_Z/A_L\right)^{(\sigma-1)/\sigma}$. In particular, substituting for (19), we have

$$\bar{\theta} = \left(\frac{\eta_Z}{\eta_L}\right)^{-\frac{\sigma-1}{1+\sigma\delta}}\left(\frac{\gamma}{1-\gamma}\right)^{\frac{\sigma-1}{1+\sigma\delta}}\left(\frac{\bar{Z}}{\bar{L}}\right)^{\frac{(\sigma-1)^2}{(1+\sigma\delta)\sigma}},$$

which is always increasing in $\bar{Z}/\bar{L}$.[20]

Next to investigate the conditions under which there is strong relative equilibrium bias, we can use Definition 6 and check the conditions for (again using log derivatives simplicity):

$$\frac{\partial \ln\left(w_Z\left(\bar{Z},\bar{L},A_Z^*,A_L^*\right)/w_L\left(\bar{Z},\bar{L},A_Z^*,A_L^*\right)\right)}{\partial \ln\left(Z/L\right)}$$
$$+\frac{\partial \ln\left(w_Z\left(\bar{Z},\bar{L},A_Z^*,A_L^*\right)/w_L\left(\bar{Z},\bar{L},A_Z^*,A_L^*\right)\right)}{\partial \ln\left(A_Z^*/A_L^*\right)}\frac{\partial \ln\left(A_Z^*/A_L^*\right)}{\partial \ln\left(Z/L\right)} > 0.$$

From (18) and (21), this condition is equivalent to

$$-\frac{1}{\sigma}+\frac{(\sigma-1)^2}{(1+\sigma\delta)\,\sigma} = \frac{\sigma-2-\delta}{1+\sigma\delta} > 0.$$

[19]The fact that $\Delta\left(w_Z/w_L\right)$ is directly a function of A_Z/A_L and expressed in terms of log changes rather than absolute changes is simply for convenience. In particular, w_Z/w_L is a function of A_Z/A_L rather than A_Z and A_L separately. Moreover, note that as long as $x>0$ and $a>0$, $\partial x/\partial a \gtreqless 0$ if and only if $\partial \ln x/\partial \ln a \gtreqless 0$. Finally, recall from footnote 17 that $\partial(Z/L)$ is used interchangeably with ∂Z.

[20]More explicitly, returning to the discussion in footnote 18, when $\sigma>1$, an increase in $\bar{Z}/\bar{L}$ increases A_Z/A_L, which in turn raises w_Z/w_L at given factor proportions. In contrast when $\sigma<1$, an increase in $\bar{Z}/\bar{L}$ reduces A_Z/A_L, but in this case, A_Z/A_L is relatively biased against factor Z (it is biased towards factor L), so a decrease in A_Z/A_L again raises w_Z/w_L.

At this point, we can also return to Hicks' quote. This example (and the theorem that follows) show that the claim in Hicks' quote is not true for relatively biased technical change; an increase in the abundance of a factor that reduces its price will make technology relatively biased towards that factor. However, if we interpret the quote in terms of relative factor-augmenting changes, equation (20) shows that it is true when $\sigma<1$. Therefore, one interpretation of Hicks' claim (though not the interpretation typically adopted in the previous literature) is that the increase in the abundance of a factor will cause technology relatively *augmenting* the other factor as long as the elasticity of substitution between the two factors is less than one.

so that when $\sigma > 2+\delta$, the relative demand curve for factors is *upward-sloping* and there is *strong relative equilibrium bias*. In this example, this result can be obtained more transparently by substituting for A_Z^*/A_L^* from (19) into (18) to obtain:

$$\frac{w_Z}{w_L} = \left(\frac{\eta_Z}{\eta_L}\right)^{-\frac{\sigma-1}{1+\sigma\delta}} \left(\frac{\gamma}{1-\gamma}\right)^{\frac{\sigma+\sigma\delta}{1+\sigma\delta}} \left(\frac{\bar{Z}}{\bar{L}}\right)^{\frac{\sigma-2-\delta}{1+\sigma\delta}}, \tag{22}$$

which implies

$$\frac{d\ln\left(w_Z\left(\bar{Z},\bar{L},A_Z^*,A_L^*\right)/w_L\left(\bar{Z},\bar{L},A_Z^*,A_L^*\right)\right)}{d\ln(Z/L)} = \frac{\sigma-2-\delta}{1+\sigma\delta},$$

which again implies that there is strong relative equilibrium bias when $\sigma > 2+\delta$.[21]

This example therefore illustrates the possibility of both weak and strong relative bias results in an economy with a non-convex aggregate production possibilities set. In particular, technological change induced in response to an increase in Z is always (weakly) relatively biased towards Z, and moreover, if the condition $\sigma > 2+\delta$ is satisfied, there is also strong relative bias. Nevertheless, the structure of the economy is quite special, in particular, it incorporates a specific aggregate production function and cost functions for undertaking research. But more important is the assumption that all technologies are assumed to be of the factor-augmenting form. I next establish that with a more general setup, but still with two-factors and factor-augmenting technologies, the same results hold.

Theorem 1 ***(Relative Equilibrium Bias with Factor-Augmenting Technologies)*** Consider Economy C or M with two-factors, $(Z,L) \in \mathcal{Z}\times\mathcal{L} \subset \mathbb{R}_+^2$, and two factor-augmenting technologies, $(A_Z, A_L) \in \mathbb{R}_+^2$, so that the production function is $F(A_Z L, A_L L)$. Assume that F is twice continuously differentiable and concave, and the costs of producing technologies are $C_Z(A_Z)$ and $C_L(A_L)$, which are also twice continuously differentiable, increasing and strictly convex. Let σ be the (local) elasticity of substitution between Z and L, defined by $\sigma = -\left.\frac{\partial\ln(Z/L)}{\partial\ln(w_Z/w_L)}\right|_{\frac{A_Z}{A_L}}$, let $\delta = \frac{\partial\ln\left(C_Z'(A_Z)/C_L'(A_L)\right)}{\partial\ln(A_Z/A_L)}$, and suppose that factor supplies are given by $\left(\bar{Z},\bar{L}\right)$. Denote equilibrium technologies by (A_Z^*,A_L^*), and equilibrium factor prices by $w_Z\left(\bar{Z},\bar{L},A_Z^*,A_L^*\right)$ and $w_L\left(\bar{Z},\bar{L},A_Z^*,A_L^*\right)$. Then we have that for all $\left(\bar{Z},\bar{L}\right) \in \mathcal{Z}\times\mathcal{L}$:

$$\frac{\partial\ln\left(A_Z^*/A_L^*\right)}{\partial\ln(Z/L)} = \frac{\sigma-1}{1+\sigma\delta} \tag{23}$$

[21] In Acemoglu (2002), the condition for upward-sloping relative demand curves was $\sigma > 2-\delta'$ for some other parameter $\delta' > 0$, which essentially corresponds to $-\delta$ here. The reason is that in that context, as in many endogenous growth models, the technology allowed for knowledge-spillovers, and the parameter δ' measured how much a particular type of technology benefits from past innovations in the same line, adding another degree of non-convexity. Here a higher value of the parameter δ makes the aggregate technology of the economy more "convex" and thus upward-sloping relative demand curves less likely. See also Theorem 8 below.

and

$$\frac{\partial \ln \left(w_Z\left(\bar{Z}, \bar{L}, A_Z^*, A_L^*\right) / w_L\left(\bar{Z}, \bar{L}, A_Z^*, A_L^*\right)\right)}{\partial \ln \left(A_Z / A_L\right)} \frac{\partial \ln \left(A_Z^* / A_L^*\right)}{\partial \ln (Z / L)} \geq 0 \tag{24}$$

so that there is always *weak relative equilibrium bias.* Moreover,

$$\frac{d \ln \left(w_Z\left(\bar{Z}, \bar{L}, A_Z^*, A_L^*\right) / w_L\left(\bar{Z}, \bar{L}, A_Z^*, A_L^*\right)\right)}{d \ln (Z / L)}=\frac{\sigma-2-\delta}{1+\sigma \delta}, \tag{25}$$

so that there is *strong relative equilibrium bias* if $\sigma-2-\delta>0$.

Proof. By Proposition 2 or 3, we need to look at the following strictly concave maximization problem:

$$\max_{A_L, A_Z} F\left(A_Z \bar{Z}, A_L \bar{L}\right)-C_Z\left(A_Z\right)-C_L\left(A_L\right).$$

Taking the ratio of the first-order necessary and sufficient conditions gives

$$\frac{\bar{Z}}{\bar{L}} \frac{F_Z\left(A_Z^* \bar{Z}, A_L^* \bar{L}\right)}{F_L\left(A_Z^* \bar{Z}, A_L^* \bar{L}\right)}=\frac{C_Z^{\prime}\left(A_Z^*\right)}{C_L^{\prime}\left(A_L^*\right)}$$

where F_Z denotes the derivative of F with respect to its first argument and F_L denotes the derivative with respect to the second. Recalling the definition of marginal products, this gives

$$\frac{\bar{Z}}{\bar{L}} \frac{w_Z\left(\bar{Z}, \bar{L}, A_Z^*, A_L^*\right)}{w_L\left(\bar{Z}, \bar{L}, A_Z^*, A_L^*\right)}=\frac{\bar{Z}}{\bar{L}} \frac{A_Z}{A_L} \frac{F_Z\left(A_Z^* \bar{Z}, A_L^* \bar{L}\right)}{F_L\left(A_Z^* \bar{Z}, A_L^* \bar{L}\right)}=\frac{A_Z}{A_L} \frac{C_Z^{\prime}\left(A_Z^*\right)}{C_L^{\prime}\left(A_L^*\right)}. \tag{26}$$

Now taking logs and differentiating totally with respect to $\ln (Z/L)$ gives:

$$\begin{aligned}\left(1+\frac{\partial \ln \left(C_Z^{\prime}\left(A_Z^*\right) / C_L^{\prime}\left(A_L^*\right)\right)}{\partial \ln \left(A_Z / A_L\right)}\right) \frac{\partial \ln \left(A_Z^* / A_L^*\right)}{\partial \ln (Z / L)} = & \left.\frac{\partial \ln \left(w_Z / w_L\right)}{\partial \ln (Z / L)}\right|_{\frac{A_Z^*}{A_L^*}}+1 \\ & +\left.\frac{\partial \ln \left(w_Z / w_L\right)}{\partial \ln \left(A_Z / A_L\right)}\right|_{\frac{Z}{L}} \frac{\partial \ln \left(A_Z^* / A_L^*\right)}{\partial \ln (Z / L)}.\end{aligned} \tag{27}$$

Equation (26) and the definition of σ yield:

$$\frac{\partial \ln \left(w_Z / w_L\right)}{\partial \ln \left(A_Z / A_L\right)}=\frac{\sigma-1}{\sigma}. \tag{28}$$

Substituting (28) into (27), rearranging and recalling the definitions of δ and σ, we obtain

$$\frac{\partial \ln \left(A_Z^* / A_L^*\right)}{\partial \ln (Z / L)}=\frac{\sigma-1}{1+\sigma \delta}$$

as in (23). Then (24) immediately follows by combining this with (28).

Finally, (25) follows from (23) by noting that

$$\frac{d \ln \left(w_Z\left(\bar{Z}, \bar{L}, A_Z^*, A_L^*\right) / w_L\left(\bar{Z}, \bar{L}, A_Z^*, A_L^*\right)\right)}{d \ln (Z / L)}=-\frac{1}{\sigma}+\frac{\sigma-1}{\sigma} \frac{\partial \ln \left(A_Z^* / A_L^*\right)}{\partial \ln (Z / L)}.$$

■

The major result of this theorem is that the insights from Example 1 generalize in a very natural way as long as the potential menu of technological possibilities only consists of factor-augmenting technologies. The only difference is that the parameter δ and the elasticity of substitution σ are no longer constants, but are functions of A_L, A_Z, $\bar{L}$ and $\bar{Z}$, so changes in factor supplies will have effects that depend on the local elasticity of substitution and the local value of δ. Nevertheless, the change in A_Z/A_L (or in $(A_Z/A_L)^{(\sigma-1)/\sigma}$ as in Example 1) induced by an increase in $\bar{Z}$ is always relatively biased towards Z, and there is strong equilibrium relative bias if $\sigma > 2 + \delta$. Therefore, this theorem establishes that an environment with a menu of technological possibilities featuring only factor-augmenting technologies is sufficient to obtain both a *general weak relative bias theorem*, and the *possibility of strong relative bias* (when the local elasticity of substitution between factors, σ, is sufficiently high and the parameter δ is relatively low).

However, once we depart from the world with only factor-augmenting technologies, it is possible for the supply of factor Z to increase, and in response, technology to change in a direction relatively biased against this factor (i.e., towards factor L), thus disproving the conjecture in the Introduction. This is stated in the next theorem and proved by providing a counterexample.

Theorem 2 With a general menu of technologies, there is not necessarily relative equilibrium bias. That is, suppose that $\partial\theta_j\left(\bar{Z},\bar{L}\right)/\partial\left(Z/L\right)$ exists at $\left(\bar{Z},\bar{L}\right)$ for all $j = 1, ..., K$, then

$$\sum_{j=1}^{K}\frac{\partial w_Z\left(\bar{Z},\bar{L},\theta\left(\bar{Z},\bar{L}\right)\right)/w_L\left(\bar{Z},\bar{L},\theta\left(\bar{Z},\bar{L}\right)\right)}{\partial\theta_j}\frac{\partial\theta_j\left(\bar{Z},\bar{L}\right)}{\partial\left(Z/L\right)}<0$$

is possible.

Example 2 ***(Counterexample)*** Consider an example of Economy C or M with the family of production functions that satisfy Assumption 2':

$$F\left(Z,L,\theta\right)=A\left(\theta\right)+\left[Z^{\theta}+L^{\theta}\right]^{1/\theta} \tag{29}$$

for $\theta \in \Theta = [a, b]$ where $b > a$, and $A(\theta)$ is concave and twice continuously differentiable over the entire Θ, with A' denoting A's first derivative.[22] From Proposition 2 or 3, the choice

[22]This way of writing the function F incorporates the cost of creating the technology, $C(\theta)$, in $A(\theta)$, which is a convenient notation I will adopt in other examples as well.

of θ will maximize $F(Z, L, \theta)$. Therefore, at given factor supplies $(\bar{Z}, \bar{L})$, the equilibrium technology choice θ satisfies

$$\frac{\partial F\left(\bar{Z}, \bar{L}, \tilde{\theta}\right)}{\partial \theta} = A'\left(\tilde{\theta}\right) - \frac{1}{\tilde{\theta}^2}\left[\bar{Z}^{\tilde{\theta}} + \bar{L}^{\tilde{\theta}}\right]^{1/\tilde{\theta}} \ln\left[\bar{Z}^{\tilde{\theta}} + \bar{L}^{\tilde{\theta}}\right] + \frac{1}{\tilde{\theta}}\left(\bar{Z}^{\tilde{\theta}} \ln \bar{Z} + \bar{L}^{\tilde{\theta}} \ln \bar{L}\right)\left[\bar{Z}^{\tilde{\theta}} + \bar{L}^{\tilde{\theta}}\right]^{(1-\tilde{\theta})/\tilde{\theta}} = 0, \tag{30}$$

with $\partial^2 F\left(\bar{Z}, \bar{L}, \tilde{\theta}\right)/\partial \theta^2 < 0$. From Definition 4, a counterexample would correspond to

$$\Delta\left(w_Z/w_L\right) \equiv \frac{\partial w_Z\left(\bar{Z}, \bar{L}, \theta\left(\bar{Z}, \bar{L}\right)\right)/w_L\left(\bar{Z}, \bar{L}, \theta\left(\bar{Z}, \bar{L}\right)\right)}{\partial \theta} \frac{\partial \theta\left(\bar{Z}, \bar{L}\right)}{\partial Z} < 0.$$

From the Implicit Function Theorem,[23] this is equivalent to

$$\Delta\left(w_Z/w_L\right) \equiv -\frac{\partial w_Z\left(\bar{Z}, \bar{L}, \theta\left(\bar{Z}, \bar{L}\right)\right)/w_L\left(\bar{Z}, \bar{L}, \theta\left(\bar{Z}, \bar{L}\right)\right)}{\partial \theta} \frac{\partial^2 F\left(\bar{Z}, \bar{L}, \theta\left(\bar{Z}, \bar{L}\right)\right)/\partial \theta \partial Z}{\partial^2 F\left(\bar{Z}, \bar{L}, \theta\left(\bar{Z}, \bar{L}\right)\right)/\partial \theta^2} < 0, \tag{31}$$

or since $\partial^2 F\left(\bar{Z}, \bar{L}, \theta\left(\bar{Z}, \bar{L}\right)\right)/\partial \theta^2 < 0$, to

$$\frac{\partial w_Z\left(\bar{Z}, \bar{L}, \theta\left(\bar{Z}, \bar{L}\right)\right)/w_L\left(\bar{Z}, \bar{L}, \theta\left(\bar{Z}, \bar{L}\right)\right)}{\partial \theta} \frac{\partial^2 F\left(\bar{Z}, \bar{L}, \theta\left(\bar{Z}, \bar{L}\right)\right)}{\partial \theta \partial Z} < 0. \tag{32}$$

To obtain this result, differentiate the expression in (30) with respect to $\bar{Z}$ and to simplify notation denote $\theta\left(\bar{Z}, \bar{L}\right)$ by $\tilde{\theta}$:

$$\begin{aligned}
\frac{\partial^2 F\left(\bar{Z}, \bar{L}, \tilde{\theta}\right)}{\partial \theta \partial Z} &= -\frac{1}{\tilde{\theta}^2} \bar{Z}^{\tilde{\theta}-1}\left[\bar{Z}^{\tilde{\theta}} + \bar{L}^{\tilde{\theta}}\right]^{\frac{1-\tilde{\theta}}{\tilde{\theta}}} \ln\left[\bar{Z}^{\tilde{\theta}} + \bar{L}^{\tilde{\theta}}\right] - \frac{1}{\tilde{\theta}} \bar{Z}^{\tilde{\theta}-1}\left[\bar{Z}^{\tilde{\theta}} + \bar{L}^{\tilde{\theta}}\right]^{\frac{1-\tilde{\theta}}{\tilde{\theta}}} \\
&+ \frac{1-\tilde{\theta}}{\tilde{\theta}} \bar{Z}^{\tilde{\theta}-1}\left(\bar{Z}^{\tilde{\theta}} \ln \bar{Z} + \bar{L}^{\tilde{\theta}} \ln \bar{L}\right)\left[\bar{Z}^{\tilde{\theta}} + \bar{L}^{\tilde{\theta}}\right]^{\frac{1-2\tilde{\theta}}{\tilde{\theta}}} + \frac{1}{\tilde{\theta}} \bar{Z}^{\tilde{\theta}-1}\left(\tilde{\theta} \ln \bar{Z} + 1\right)\left[\bar{Z}^{\tilde{\theta}} + \bar{L}^{\tilde{\theta}}\right]^{\frac{1-\tilde{\theta}}{\tilde{\theta}}} \\
&\propto -\frac{1}{\tilde{\theta}} \ln\left[\bar{Z}^{\tilde{\theta}} + \bar{L}^{\tilde{\theta}}\right] - 1 + \left(1-\tilde{\theta}\right)\left(\bar{Z}^{\tilde{\theta}} \ln \bar{Z} + \bar{L}^{\tilde{\theta}} \ln \bar{L}\right)\left[\bar{Z}^{\tilde{\theta}} + \bar{L}^{\tilde{\theta}}\right]^{-1} + \left(\tilde{\theta} \ln \bar{Z} + 1\right)
\end{aligned}$$

If this expression is negative, then in response to an increase in $\bar{Z}$ (or $\bar{Z}/\bar{L}$), $\tilde{\theta} = \theta\left(\bar{Z}, \bar{L}\right)$ will decline. Moreover, from (29), we have

$$\frac{w_Z\left(Z, L, \tilde{\theta}\right)}{w_L\left(Z, L, \tilde{\theta}\right)} = \left(\frac{Z}{L}\right)^{\tilde{\theta}-1},$$

which is increasing in $\tilde{\theta}$ as long as $Z > L$. Now suppose that we start with $\bar{L} = 1$ and $\bar{Z} = 2$, and choose the function $A(\theta)$ such that $\tilde{\theta} = 0.1$. Then

$$\begin{aligned}
\frac{\partial^2 F\left(\bar{Z} = 2, \bar{L} = 1, \tilde{\theta} = 0.1\right)}{\partial \theta \partial Z} &\propto -\frac{1}{0.1} \ln\left[1 + 2^{0.1}\right] - 1 + 0.9\left(2^{0.1} \ln 2\right)\left[1 + 2^{0.1}\right]^{-1} + (0.1 \ln 2 + 1) \\
&\propto -7.28 - 1 + 0.32 + 1.07 < 0,
\end{aligned}$$

[23] See, for example, Rudin (1964), Theorem 9.18, or Simon and Blume (1994), Theorem 15.2.

which is clearly negative, so

$$\frac{\partial w_Z\left(\bar{Z}=2,\bar{L}=1,\tilde{\theta}=0.1\right)/w_L\left(\bar{Z}=2,\bar{L}=1,\tilde{\theta}=0.1\right)}{\partial\theta}\times\frac{\partial^2 F\left(\bar{Z}=2,\bar{L}=1,\tilde{\theta}=0.1\right)}{\partial\theta\partial Z}<0,$$

and (32) is satisfied, providing a counterexample to the conjecture. Put differently, in this case the increase in $\bar{Z}$ induces a decline in $\tilde{\theta}$, which is a change in technology relatively biased against Z.

Theorem 1 explains the reason for the negative result in Theorem 2. The conjecture about relative bias does not apply in this example because technologies do not take the factor-augmenting form. Although factor-augmenting technology may be an interesting and empirically important special case, one may be interested in a more general theorem that applies without imposing a specific structure on the interaction between technologies and the factors of production. This is especially the case when we consider technology choices that correspond to shifts from one type of organizational form or organization of production to another, such as those experienced during recent decades, during the emergence of the American System of Manufacturing, or during the Industrial Revolution. These shifts not only change the productivity of different factors, but the way the whole production process is organized and thus naturally also the elasticity of substitution between factors.

Example 2 shows that a general theorem is not possible for relative bias. In fact, Example 2 shows a very simple case where a choice over technologies with different elasticities of substitution can reverse the results in Theorem 1. Essentially, once we expand the set of technologies to allow for non-factor-augmenting ones, this will always effectively amount to a choice between technologies with different elasticities of substitution, so this example clarifies the limits of Theorem 1. Nevertheless, it is also important to emphasize that this example and Theorem 2 do *not* imply that with the general menu of technologies, changes in relative supplies will cause technical change that it is relatively biased against the more abundant factor. In many cases, weak equilibrium bias will still apply, but without imposing more structure, we do not have a general theorem.

In the next section, we will see that such a theorem can be derived for absolute bias. In fact, Example 2 already hints at this possibility. The reason why induced technology (in response to an increase in $\bar{Z}$) is not relatively biased towards Z is that the induced change in technology affects the elasticity of substitution between the two factors, and consequently, even though it increases w_Z (at given factor proportions), it has an even larger (positive) effect on the marginal product of the other factor, w_L.[24]

4 Equilibrium Absolute Bias

Example 2 shows that there is no general theorem about relative equilibrium bias unless we restrict ourselves to factor-augmenting technologies. The obvious question is whether there is a general result for absolute bias. The answer is yes and is the focus of this section. Recall that *absolute bias* refers to whether new technology increases the marginal product of a factor. The main results in this section will therefore show that in response to increases in the supply of a factor (or a set of factors), technology will change endogenously in a direction absolutely biased towards this factor (or this set of factors).

As stated in the Introduction, this section focuses on weak (absolute) bias results and presents both local and global theorems. I begin with the local theorem, which applies to the case with $N = 1$, i.e., to changes in the supply of a single factor, Z.

Given the results in Section 2, the problem of equilibrium technology choice is again equivalent to

$$\max_{\theta\in\Theta} F\left(\bar{Z}, \bar{\mathbf{L}}, \theta\right) \tag{33}$$

where $\bar{\mathbf{L}}$ denotes the supply of these other inputs and $\bar{Z}$ denotes the supply of Z. Let us denote the marginal product (or price) of this factor by $w_Z\left(\bar{Z}, \bar{\mathbf{L}}, \theta\right) = \partial F\left(\bar{Z}, \bar{\mathbf{L}}, \theta\right)/\partial Z$ when the employment levels of factors are given by $\left(\bar{Z}, \bar{\mathbf{L}}\right)$ and the technology is θ. For the local result I will also take Θ to be a convex compact subset of $\mathbb{R}^K$ for some $K \geq 1$ and assume that F is also twice differentiable in (Z, θ), which implies that $w_Z\left(\bar{Z}, \bar{\mathbf{L}}, \theta\right)$ is differentiable in θ.

Definition 7 Let $\theta \in \Theta \subset \mathbb{R}^K$. An increase in technology θ_j for some $j = 1, ..., K$ is *locally absolutely biased* towards factor Z at $\left(\bar{Z}, \bar{\mathbf{L}}\right) \in \mathcal{Z} \times \mathcal{L}$ if

$$\frac{\partial w_Z\left(\bar{Z}, \bar{\mathbf{L}}, \theta\right)}{\partial \theta_j} \geq 0.$$

[24] To see this more explicitly, note that $\partial^2 F\left(\bar{Z}, \bar{L}, \theta\left(\bar{Z}, \bar{L}\right)\right)/\partial\theta\partial Z = \partial w_Z\left(\bar{Z}, \bar{L}, \theta\left(\bar{Z}, \bar{L}\right)\right)/\partial\theta$ and rewrite equation (31) as

$$\Delta\left(w_Z/w_L\right) \equiv -\frac{\partial w_Z\left(\bar{Z}, \bar{L}, \theta\left(\bar{Z}, \bar{L}\right)\right)/w_L\left(\bar{Z}, \bar{L}, \theta\left(\bar{Z}, \bar{L}\right)\right)}{\partial\theta}\frac{\partial w_Z\left(\bar{Z}, \bar{L}, \theta\left(\bar{Z}, \bar{L}\right)\right)/\partial\theta}{\partial^2 F\left(\bar{Z}, \bar{L}, \theta\left(\bar{Z}, \bar{L}\right)\right)/\partial\theta^2}.$$

When $w_L\left(\bar{Z}, \bar{L}, \theta\left(\bar{Z}, \bar{L}\right)\right)$ is constant, this is equivalent to equation (39) in the proof of Theorem 3 in the next section and is always nonnegative. However, as Example 2 shows, a large effect of θ on w_L can reverse this result.

Conversely we could define a decrease in technology θ as locally absolutely biased towards factor Z if the same derivative is nonpositive. Notice also that the local bias definition requires the bias for only small changes in technology and only at the *current* factor proportions $\left(\bar{Z}, \bar{\mathbf{L}}\right)$. The global definition below will require a similar directional change but for *all* magnitudes of changes in supplies and at *all* factor proportions. Next we define (local) equilibrium absolute bias analogously to relative equilibrium bias.

Definition 8 Let $\theta \in \Theta \subset \mathbb{R}^K$. Denote the equilibrium technology at factor supplies $\left(\bar{Z}, \bar{\mathbf{L}}\right) \in \mathcal{Z} \times \mathcal{L}$ by $\theta\left(\bar{Z}, \bar{\mathbf{L}}\right)$ and assume that $\partial\theta_j\left(\bar{Z}, \bar{\mathbf{L}}\right)/\partial Z$ exists at $\left(\bar{Z}, \bar{\mathbf{L}}\right)$ for all $j = 1, ..., K$. Then there is *local absolute equilibrium bias* at $\left(\bar{Z}, \bar{\mathbf{L}}, \theta\left(\bar{Z}, \bar{\mathbf{L}}\right)\right)$ if

$$\sum_{j=1}^{K} \frac{\partial w_Z\left(\bar{Z}, \bar{\mathbf{L}}, \theta\left(\bar{Z}, \bar{\mathbf{L}}\right)\right)}{\partial \theta_j} \frac{\partial \theta_j\left(\bar{Z}, \bar{\mathbf{L}}\right)}{\partial Z} \geq 0. \tag{34}$$

In words, this definition requires the induced combined change in the components of technology resulting from an increase in $\bar{Z}$ to be towards increasing the marginal product of factor Z. As in Definition 5 for relative equilibrium bias, this definition also requires $\partial\theta_j\left(\bar{Z}, \bar{\mathbf{L}}\right)/\partial Z$ to exist for all j. The next theorem will also be stated under this assumption, which can alternatively be replaced by Assumption A1 below.

Theorem 3 *(Local Absolute Bias)* Consider Economy D, C or M. Suppose that Θ is a convex subset of $\mathbb{R}^K$ and $F(Z, \mathbf{L}, \theta)$ is twice continuously differentiable in (Z, θ). Let the equilibrium technology at factor supplies $\left(\bar{Z}, \bar{\mathbf{L}}\right)$ be $\theta\left(\bar{Z}, \bar{\mathbf{L}}\right)$, and assume that $\theta\left(\bar{Z}, \bar{\mathbf{L}}\right)$ is in the interior of Θ and that $\partial\theta_j\left(\bar{Z}, \bar{\mathbf{L}}\right)/\partial Z$ exists at $\left(\bar{Z}, \bar{\mathbf{L}}\right)$ for all $j = 1, ..., K$. Then, there is *local absolute equilibrium bias* at all $\left(\bar{Z}, \bar{\mathbf{L}}\right) \in \mathcal{Z} \times \mathcal{L}$, i.e.,

$$\sum_{j=1}^{K} \frac{\partial w_Z\left(\bar{Z}, \bar{\mathbf{L}}, \theta\left(\bar{Z}, \bar{\mathbf{L}}\right)\right)}{\partial \theta_j} \frac{\partial \theta_j\left(\bar{Z}, \bar{\mathbf{L}}\right)}{\partial Z} \geq 0 \text{ for all } \left(\bar{Z}, \bar{\mathbf{L}}\right) \in \mathcal{Z} \times \mathcal{L}. \tag{35}$$

Moreover, if $\partial\theta_j\left(\bar{Z}, \bar{\mathbf{L}}\right)/\partial Z \neq 0$ for some $j = 1, ..., K$, then

$$\sum_{j=1}^{K} \frac{\partial w_Z\left(\bar{Z}, \bar{\mathbf{L}}, \theta\left(\bar{Z}, \bar{\mathbf{L}}\right)\right)}{\partial \theta_j} \frac{\partial \theta_j\left(\bar{Z}, \bar{\mathbf{L}}\right)}{\partial Z} > 0. \tag{36}$$

Proof. The proof follows from the Implicit Function Theorem. For expositional clarity, I first present the case where $\theta \in \Theta \subset \mathbb{R}$. Since $\Theta \subset \mathbb{R}$ and by hypothesis, the equilibrium choice of θ is in the interior of Θ, we have

$$\frac{\partial F\left(\bar{Z}, \bar{\mathbf{L}}, \theta\left(\bar{Z}, \bar{\mathbf{L}}\right)\right)}{\partial \theta} = 0, \tag{37}$$

and $\partial^2 F\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)/\partial\theta^2 \leq 0$. Since $\partial\theta\left(\bar{Z},\bar{\mathbf{L}}\right)/\partial Z$ exists at $\left(\bar{Z},\bar{\mathbf{L}}\right)$ by hypothesis, from the Implicit Function Theorem it must be equal to

$$\frac{\partial\theta\left(\bar{Z},\bar{\mathbf{L}}\right)}{\partial Z} = -\frac{\partial^2 F\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)/\partial\theta\partial Z}{\partial^2 F\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)/\partial\theta^2} = -\frac{\partial w_Z\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)/\partial\theta}{\partial^2 F\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)/\partial\theta^2}, \tag{38}$$

so we must have $\partial^2 F\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)/\partial\theta^2 \neq 0$, i.e., $\partial^2 F\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)/\partial\theta^2 < 0$. This in turn implies:

$$\frac{\partial w_Z\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)}{\partial\theta}\frac{\partial\theta\left(\bar{Z},\bar{\mathbf{L}}\right)}{\partial Z} = -\frac{\left[\partial w_Z\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)/\partial\theta\right]^2}{\partial^2 F\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)/\partial\theta^2} \geq 0, \tag{39}$$

establishing (35). Moreover, if $\partial\theta\left(\bar{Z},\bar{\mathbf{L}}\right)/\partial Z \neq 0$, then from (38), $\partial w_Z\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)/\partial\theta \neq 0$, so (39) holds with strict inequality, establishing (36).

Next, let us look at the general case where $\theta \in \Theta \subset \mathbb{R}^K$ with $K > 1$. Let Δw_Z be the change in w_Z resulting from the induced change in θ (at given factor proportions) as in equation (34):

$$\Delta w_Z = \sum_{j=1}^{K}\frac{\partial w_Z\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)}{\partial\theta_j}\frac{\partial\theta_j\left(\bar{Z},\bar{\mathbf{L}}\right)}{\partial Z}.$$

Then, we have that

$$\begin{aligned}\Delta w_Z &= \left[\nabla_\theta w_Z\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)\right]'\left[\nabla_Z\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right] \\ &= \left[\nabla^2_{\theta Z}F\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)\right]'\left[\nabla_Z\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right],\end{aligned} \tag{40}$$

where $\left[\nabla_\theta w_Z\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)\right]$ is a $K\times 1$ vector of changes in w_Z in response to each component of $\theta \in \Theta \subset \mathbb{R}^K$, $\left[\nabla_Z\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right]$ is the Jacobian of θ with respect to Z, i.e., a $K \times 1$ vector of changes in each component of θ in response to the change in $\bar{Z}$, and for a matrix (vector) v, v' denotes its transpose. The second line in (40) uses the fact that w_Z is the derivative of the F function, so $\left[\nabla^2_{\theta Z}F\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)\right]$ is also the $K \times 1$ vector of changes in w_Z in response to each component of θ. Since $\partial\theta_j\left(\bar{Z},\bar{\mathbf{L}}\right)/\partial Z$ exists at $\left(\bar{Z},\bar{\mathbf{L}}\right)$ for all j, the gradient $\nabla_Z\theta\left(\bar{Z},\bar{\mathbf{L}}\right)$ also exists and from the Implicit Function Theorem, it satisfies

$$\nabla_Z\theta\left(\bar{Z},\bar{\mathbf{L}}\right)' = -\left[\nabla^2_{\theta Z}F\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)\right]'\left[\nabla^2_{\theta\theta}F\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)\right]^{-1},$$

where $\nabla^2_{\theta\theta}F\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)$ is the $K \times K$ Hessian of F with respect to θ. The fact that $\theta\left(\bar{Z},\bar{\mathbf{L}}\right)$ is a solution to the maximization problem (33) implies that $\nabla^2_{\theta\theta}F\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)$ is negative semi-definite. That $\nabla_Z\theta\left(\bar{Z},\bar{\mathbf{L}}\right)$ exists then implies that $\nabla^2_{\theta\theta}F\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)$ is non-singular, so it must be negative definite. Since it is a Hessian, it is also symmetric. Therefore, its inverse $\left[\nabla^2_{\theta\theta} F\left(\bar{Z}, \bar{\mathbf{L}}, \theta\left(\bar{Z}, \bar{\mathbf{L}}\right)\right)\right]^{-1}$ is also symmetric and negative definite. Substituting in (40), we obtain

$$\Delta w_Z = -\left[\nabla^2_{\theta Z} F\left(\bar{Z}, \bar{\mathbf{L}}, \theta\left(\bar{Z}, \bar{\mathbf{L}}\right)\right)\right]' \left[\nabla^2_{\theta\theta} F\left(\bar{Z}, \bar{\mathbf{L}}, \theta\left(\bar{Z}, \bar{\mathbf{L}}\right)\right)\right]^{-1} \left[\nabla^2_{\theta Z} F\left(\bar{Z}, \bar{\mathbf{L}}, \theta\left(\bar{Z}, \bar{\mathbf{L}}\right)\right)\right] \geq 0,$$

which establishes (35) for the case in which $\Theta \subset \mathbb{R}^K$.

By the definition of a negative definite matrix B, $x'Bx < 0$ for all $x \neq 0$, so to establish the strict inequality in (36) in this case, it suffices that one element of $\nabla_Z \theta\left(\bar{Z}, \bar{\mathbf{L}}\right)$ be non-zero, i.e., $\partial\theta_j\left(\bar{Z}, \bar{\mathbf{L}}\right)/\partial Z \neq 0$ for one $j = 1, ...K$. ■

This theorem therefore shows that once we shift our focus to absolute bias, there is a fairly general result. Under minimal assumptions (further discussed below), technological change induced by a change in factor supplies will be biased towards the factor that has become more abundant. There is a clear parallel here with the LeChatelier principle of Samuelson (1947), but also a number of important differences. First, this theorem concerns how marginal products (or prices) change as a result of induced technological changes resulting from changes in factor supplies rather than the elasticity of short-run and long-run demand curves. Second, it applies to the equilibrium of an economy not to the maximization problem of a single firm. Nevertheless, the parallel is also important, since we can think of the change in technology as happening in the "long run", in which case Theorem 3 states that long-run changes in marginal products (factor prices) will be less than those in the short run because of induced technological change or technology adoption.

Theorem 3 was stated and proved under the assumption that $\partial\theta_j\left(\bar{Z}, \bar{\mathbf{L}}\right)/\partial Z$ exists at $\left(\bar{Z}, \bar{\mathbf{L}}\right)$ for all $j = 1, ..., K$. This assumption entails two restrictions. The first is the usual non-singularity requirement to enable an application of the Implicit Function Theorem, i.e., that the Hessian of F with respect to θ, $\nabla^2_{\theta\theta} F\left(\bar{Z}, \bar{L}, \theta\left(\bar{Z}, \bar{L}\right)\right)$, is non-singular at the point $\theta\left(\bar{Z}, \bar{L}\right)$ (see, for example, Rudin, 1964, Theorem 9.18, or Simon and Blume, 1994, Theorem 15.2; recall also the argument in the proof of Theorem 3). The second is more subtle; since we have not made global concavity assumptions (except in Economy D), a small change in Z may shift the technology choice from one local optimum to another, thus essentially making $\partial\theta_j\left(\bar{Z}, \bar{L}\right)/\partial Z$ infinite (or undefined). This possibility is also ruled out by this assumption. In fact, the assumption that $\partial\theta_j\left(\bar{Z}, \bar{L}\right)/\partial Z$ exists at $\left(\bar{Z}, \bar{L}\right)$ can be replaced by the following:

Assumption A1: $\nabla^2_{\theta\theta} F\left(\bar{Z}, \bar{L}, \theta\left(\bar{Z}, \bar{L}\right)\right)$ is non-singular, and there exists $\xi > 0$ such that for all $\theta' \in \Theta$ with $\partial F\left(\bar{Z}, \bar{L}, \theta'\right)/\partial\theta = 0$ and $\theta' \neq \theta\left(\bar{Z}, \bar{L}\right)$, we have $F\left(\bar{Z}, \bar{L}, \theta\left(\bar{Z}, \bar{L}\right)\right) - F\left(\bar{Z}, \bar{L}, \theta'\right) > \xi$.

The second part of the assumption ensures that the peaks of the function $F(\bar{Z}, \bar{L}, \theta)$ in θ are "well separated", in the sense that in response to a small change in factor supplies, there will not be a shift in the global optimum of θ from one local optimum to another.[25] Consequently, Assumption A1 is equivalent to assuming that $\partial\theta_j(\bar{Z}, \bar{L})/\partial Z$ exists at $(\bar{Z}, \bar{L})$ for all j. A straightforward condition to ensure that Assumption A1 is satisfied is to assume that F is strictly quasi-concave in θ, though this is considerably stronger than Assumption A1. Since it is more intuitive to directly assume that the derivatives $\partial\theta_j(\bar{Z}, \bar{\mathbf{L}})/\partial Z$'s exist rather than imposing Assumption A1, I state the relevant theorems under this direct assumption. But depending on taste, Assumption A1 can be substituted in Theorem 3 and some of the subsequent theorems.

Three shortcomings of Theorem 3 are apparent. First, it applies to changes in the supply of a single factor. Second, it applies only to local (small) changes. Third, and perhaps most importantly, as Definition 8 makes it clear, equilibrium bias is a local phenomenon. For example, an increase in $\bar{Z}$ may change $\theta(\bar{Z}, \bar{\mathbf{L}})$ in a direction biased towards Z at factor proportions $(\bar{Z}, \bar{\mathbf{L}})$, but this change may in fact be biased against Z at some different factor proportions, say $(\bar{Z}', \bar{\mathbf{L}}')$. Similar to Milgrom and Roberts' (1996) generalization of LeChatelier principle, there is a global version of Theorem 3, which will strengthen and generalize it to deal with all of these problems.[26] Like the results in Milgrom and Roberts (1996), this theorem also uses tools from the theory of monotone comparative statics. I start with changes in a single factor, and then generalize it to multiple factors.

Definition 9 Let $\theta(\bar{Z}, \bar{\mathbf{L}})$ be the equilibrium technology choice in an economy with factor supplies $(\bar{Z}, \bar{\mathbf{L}})$. We say that there is *global absolute equilibrium* bias if for any $\bar{Z}', \bar{Z} \in \mathcal{Z}$,

$$\bar{Z}' \geq \bar{Z} \Longrightarrow w_Z\left(\tilde{Z}, \bar{\mathbf{L}}, \theta\left(\bar{Z}', \bar{\mathbf{L}}\right)\right) \geq w_Z\left(\tilde{Z}, \bar{\mathbf{L}}, \theta\left(\bar{Z}, \bar{\mathbf{L}}\right)\right) \text{ for all } \tilde{Z} \in \mathcal{Z} \text{ and } \bar{\mathbf{L}} \in \mathcal{L}.$$

[25] Put differently, suppose that the equilibrium maximization problem (33) has multiple local maxima, and denote the set of these maxima at factor proportions $(\bar{Z}, \bar{L})$ by $\Theta^s(\bar{Z}, \bar{L})$. All of these solutions satisfy the first-order necessary conditions of the equilibrium maximization problem (33). Suppose $\hat{\theta}(\bar{Z}, \bar{L})$ is a vector that satisfies these first-order necessary conditions. Given the non-singularity assumption (first part of Assumption A1), the Implicit Function Theorem can be applied to $\hat{\theta}(\bar{Z}, \bar{L})$. However, this does not guarantee that $\partial\theta(\bar{Z}, \bar{L})/\partial Z$ exists, since $\theta(\bar{Z}, \bar{L})$ corresponds to the global maximum of (33), and the change in Z may shift the global maximum from $\hat{\theta}(\bar{Z}, \bar{L})$ to some other $\tilde{\theta}(\bar{Z}, \bar{L}) \in \Theta^s(\bar{Z}, \bar{L})$. The second part of Assumption A1 rules this possibility out by imposing that one of the solutions to the first-order necessary conditions gives *uniformly* higher value, so that a small (infinitesimal) change in Z cannot induce a shift from one element of $\Theta^s(\bar{Z}, \bar{L})$ to another.

[26] A fourth potential shortcoming is that Theorem 3 is stated assuming that $\theta(\bar{Z}, \bar{\mathbf{L}})$ is in the interior of Θ. This is straightforward to relax. Nevertheless, since the global theorem, Theorem 4, naturally covers the case in which $\theta(\bar{Z}, \bar{\mathbf{L}})$ may be at the boundary of Θ, I do not introduce the additional notation to deal with this case in Theorem 3 (see also the proof of Theorem 6).

Note that there are two notions of "globality" in this definition. First, the increase from $\bar{Z}$ to $\bar{Z}'$ is not limited to small changes. Second, the change in technology induced by this increase is required to increase the price of factor Z for all $\tilde{Z} \in \mathcal{Z}$. Once again, this definition can be made stronger by requiring strict inequality.

To state the main results, we need a number of technical definitions. Appendix B defines lattices, supermodular functions and (strictly) increasing differences. Both supermodularity and increasing differences loosely correspond to the notion of complementarities, i.e., the requirement that a change in a function resulting from an increase in one argument is itself increasing in the other arguments. In the case of continuously differentiable functions, we have a particularly simple definition of supermodularity (see, e.g., Topkis, 1998):

Definition 10 Let $x = (x_1, ..., x_n)$ be a vector in $X \subset \mathbb{R}^n$, and suppose that the real-valued function $f(x)$ is twice continuously differentiable in x. Then $f(x)$ is supermodular on X if and only if $\partial^2 f(x) / \partial x_i \partial x_{i'} \geq 0$ for all $x \in X$ and for all $i \neq i'$.

Increasing differences is a slightly weaker concept again related to complementarities:

Definition 11 Let X and T be partially ordered sets. Then a function $f(x,t)$ defined on a subset S of $X \times T$ has *increasing differences in* (x,t), if for all $t'' > t$, $f(x,t'') - f(x,t)$ is nondecreasing in x. Moreover, $f(x,t)$ has *strictly increasing differences in* (x,t), if for all $t'' > t$, $f(x,t'') - f(x,t)$ is strictly increasing in x.

Lemma 2 in Appendix B shows that (strict) supermodularity in (x,t) implies (strict) increasing differences in (x,t). With these definitions, we can use Topkis' Monotonicity Theorem, given as Theorem 10 in Appendix B. Using this approach, we now have:

Theorem 4 ***(Global Equilibrium Bias)*** Consider Economy D, C or M. Suppose that Θ is a lattice, let $\bar{\mathcal{Z}}$ be the convex hull of $\mathcal{Z}$, let $\theta(\bar{Z}, \bar{\mathbf{L}})$ be the equilibrium technology at factor proportions $(\bar{Z}, \bar{\mathbf{L}})$, and suppose that $F(Z, \mathbf{L}, \theta)$ is continuously differentiable in Z, supermodular in θ on Θ for all $Z \in \bar{\mathcal{Z}}$ and $\mathbf{L} \in \mathcal{L}$, and exhibits strictly increasing differences in (Z, θ) on $\bar{\mathcal{Z}} \times \Theta$ for all $\mathbf{L} \in \mathcal{L}$, then there is *global absolute equilibrium bias*, i.e., for any $\bar{Z}', \bar{Z} \in \mathcal{Z}$, $\bar{Z}' \geq \bar{Z}$ implies

$$\theta(\bar{Z}', \bar{\mathbf{L}}) \geq \theta(\bar{Z}, \bar{\mathbf{L}}) \text{ for all } \bar{\mathbf{L}} \in \mathcal{L}$$

and

$$w_Z\left(\tilde{Z}, \bar{\mathbf{L}}, \theta(\bar{Z}', \bar{\mathbf{L}})\right) \geq w_Z\left(\tilde{Z}, \bar{\mathbf{L}}, \theta(\bar{Z}, \bar{\mathbf{L}})\right) \text{ for all } \tilde{Z} \in \mathcal{Z} \text{ and } \bar{\mathbf{L}} \in \mathcal{L}.$$

Proof. The proof follows from the application of Theorem 10 in Appendix B. Given the continuity and the supermodularity of $F(Z, \mathbf{L}, \theta)$ on $\bar{\mathcal{Z}} \times \Theta$ and the fact that Θ is a lattice and $\mathcal{Z}$ is a subset of $\mathbb{R}$ therefore also a lattice, Theorem 10 in Appendix B implies that the set of equilibrium technologies is a non-empty, compact and complete sublattice of Θ. Moreover, supermodularity of F in θ and strict increasing differences in (Z, θ) implies that $\bar{Z}' \geq \bar{Z} \Longrightarrow \theta\left(\bar{Z}', \bar{\mathbf{L}}\right) \geq \theta\left(\bar{Z}, \bar{\mathbf{L}}\right)$ for all $\bar{\mathbf{L}} \in \mathcal{L}$. Next (strict) increasing differences of $F(Z, \mathbf{L}, \theta)$ in (Z, θ) on $\bar{\mathcal{Z}} \times \Theta$ implies that $\partial F\left(\tilde{Z}, \bar{\mathbf{L}}, \theta\right) / \partial Z$ is increasing in θ for all $\tilde{Z} \in \left[\bar{Z}, \bar{Z}'\right] \subset \bar{\mathcal{Z}}$. Since $w_Z\left(\tilde{Z}, \bar{\mathbf{L}}, \theta\left(\bar{Z}', \bar{\mathbf{L}}\right)\right) = \partial F\left(\tilde{Z}, \bar{\mathbf{L}}, \theta\left(\bar{Z}', \bar{\mathbf{L}}\right)\right) / \partial Z$, the conclusion follows. ■

An important feature of this theorem, as opposed to the local theorem, Theorem 3, is that consistent with Definition 9, the induced change in technology does not only increase the marginal product of factor Z (which is becoming more abundant) at the current supply, $\bar{Z}$, but does so at all points in the set $\mathcal{Z}$. In this sense, Theorem 4 is indeed a global theorem, applying both for large magnitudes of changes and applying to all admissible levels of factor supplies.[27]

Also in this theorem, the fact that $\theta\left(\bar{Z}', \bar{\mathbf{L}}\right) \geq \theta\left(\bar{Z}, \bar{\mathbf{L}}\right)$ (say rather than $\theta\left(\bar{Z}', \bar{\mathbf{L}}\right) \leq \theta\left(\bar{Z}, \bar{\mathbf{L}}\right)$) is not particularly important, since the order over the set Θ is not specified. It could be that as $\bar{Z}$ increases some measure of technology t declines. But in this case, this measure would correspond to a type of technology biased *against* factor Z. If so, we can simply change the order over this parameter, e.g., we can consider changes in $\tilde{t} = -t$ rather than t.

Remark 1 Inspection of Theorem 10 in Appendix B will show that Theorem 4 also applies, when the assumption that F is supermodular in θ is replaced with the weaker assumption that F is quasi-supermodular in θ, which is an ordinal property introduced by Milgrom and Shannon (1994). But Example 3 below shows that (strict) increasing differences cannot be replaced with the (strict) single crossing property of Milgrom and Shannon (1994). Thus, interestingly, Theorem 4 requires a mixture of ordinal and cardinal conditions. Nevertheless, I stated the result under the stronger assumption of supermodularity since quasi-supermodularity is only defined in Appendix B.

Remark 2 Theorem 4 can also be stated assuming only increasing differences in (Z, θ) rather than strict increasing differences. But in this case, the conclusion that $\theta\left(\bar{Z}', \bar{\mathbf{L}}\right) \geq \theta\left(\bar{Z}, \bar{\mathbf{L}}\right)$ would only apply to the greatest and the least elements of the set of equilibrium technologies

[27] It is also useful to note that Theorem 3 could be derived from Theorem 4 by restricting the set Θ to an arbitrarily small ball around $\theta\left(\bar{Z}, \bar{\mathbf{L}}\right)$ and then taking a second-order Taylor approximation to F as in Corollary to Theorem 2 of Milgrom and Roberts (1996, p. 176).

(see Theorem 2.8.1 in Topkis, 1998). The greatest and the least elements of the maximizer set always exist since, given the assumptions of Theorem 4, Theorem 10 ensures that the maximizer set is a complete lattice. Given this result, in the discussion I will simplify the terminology by often referring to increasing differences rather than strict increasing differences.

Two immediate related corollaries of this theorem are also useful to note. Both of those strengthen the results of the theorem to obtain strict inequalities. The first states that this is the case whenever $\theta\left(\bar{Z}',\bar{\mathbf{L}}\right) > \theta\left(\bar{Z},\bar{\mathbf{L}}\right)$ (so it is similar to the result on strict inequalities in Theorem 3), while the second imposes additional conditions to ensure $\theta\left(\bar{Z}',\bar{\mathbf{L}}\right) > \theta\left(\bar{Z},\bar{\mathbf{L}}\right)$.

Corollary 1 Suppose that the hypotheses in Theorem 4 hold. If in addition $\theta\left(\bar{Z}',\bar{\mathbf{L}}\right) \neq \theta\left(\bar{Z},\bar{\mathbf{L}}\right)$, we have $w_Z\left(\tilde{Z},\bar{\mathbf{L}},\theta\left(\bar{Z}',\bar{\mathbf{L}}\right)\right) > w_Z\left(\tilde{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)$ for all $\tilde{Z} \in \mathcal{Z}$ and $\bar{\mathbf{L}} \in \mathcal{L}$.

Proof. Since $w_Z\left(\tilde{Z},\bar{\mathbf{L}},\theta\left(\bar{Z}',\bar{\mathbf{L}}\right)\right) = \partial F\left(\tilde{Z},\bar{\mathbf{L}},\theta\left(\bar{Z}',\bar{\mathbf{L}}\right)\right)/\partial Z$, $\theta\left(\bar{Z}',\bar{\mathbf{L}}\right) \geq \theta\left(\bar{Z},\bar{\mathbf{L}}\right)$ and $\theta\left(\bar{Z}',\bar{\mathbf{L}}\right) \neq \theta\left(\bar{Z},\bar{\mathbf{L}}\right)$ combined with the fact that F exhibits strict increasing differences in (Z,θ) establish this result. ■

Corollary 2 Suppose that the hypotheses in Theorem 4 hold. Suppose in addition that Θ is a convex compact subset of $\mathbb{R}^K$ and that F is twice continuously differentiable $(Z,\theta,\mathbf{L})$ and $\partial F\left(\tilde{Z},\bar{\mathbf{L}},\theta\right)/\partial Z$ is strictly increasing in θ for all $\tilde{Z} \in Z$. Consider any $\bar{Z}',\bar{Z} \in Z$, such that $\bar{Z}' > \bar{Z}$ and $\bar{\mathbf{L}} \in \mathcal{L}$, and suppose that $\theta\left(\bar{Z}',\bar{\mathbf{L}}\right)$ and $\theta\left(\bar{Z},\bar{\mathbf{L}}\right)$ are in the interior of Θ. Then, we have that $\theta\left(\bar{Z}',\bar{\mathbf{L}}\right) > \theta\left(\bar{Z},\bar{\mathbf{L}}\right)$ and $w_Z\left(\tilde{Z},\bar{\mathbf{L}},\theta\left(\bar{Z}',\bar{\mathbf{L}}\right)\right) > w_Z\left(\tilde{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)$.

Proof. This corollary follows immediately from the Strong Monotonicity Theorem, Theorem 11, in Appendix B. ■

An important implication of Theorem 4 and the subsequent corollaries is that we now have a global version of Theorem 3, but at the expense of introducing more structure. In particular, in addition to the relatively weak assumptions (in this context) that $\mathcal{Z}$ and Θ are lattices, we need F to be (quasi-)supermodular in θ and to exhibit (strict) increasing differences in (Z,θ).

More importantly, there are limits to how much Theorems 3 and 4 can be generalized. *First,* Theorem 3 does not apply for large changes in Z. In fact, quite interestingly, we cannot take a series of small changes and turn them into a biased response for a large change in Z (without the additional supermodularity and increasing differences assumption). *Second,* the requirement of (strict) increasing differences in (Z,θ) in Theorem 4 cannot be dispensed with, nor could it be replaced by the weaker conditions of single-crossing or quasi-supermodularity in

(Z,θ) of Milgrom and Shannon (1994)—see Appendix B for definitions. *Third*, the assumption that F should exhibit increasing differences on the convex hull of $\mathcal{Z}$ rather than $\mathcal{Z}$ itself cannot be dispensed with either. The following example illustrates all these features by constructing a simple economy which satisfies the local theorem, Theorem 3, at all points, but fails to yield a global version of (absolute) bias because the production function does not exhibit increasing differences in (Z,θ) on $\bar{\mathcal{Z}} \times \Theta$ (though it satisfies single crossing in (Z,θ) and in (θ,Z), and quasi-supermodularity in (Z,θ) and exhibits increasing differences on $\mathcal{Z} \times \Theta$ where $\mathcal{Z}$ is nonconvex).

Example 3 *(No Global Bias Without Increasing Differences)* Suppose that $F(Z,\mathbf{L},\theta) = Z + \left(Z - Z^2/8\right)\theta + A(\theta) + B(\mathbf{L})$ and $Z \in \mathcal{Z} = [0,6]$ and $\Theta = [0,2]$ so that F is everywhere increasing in Z. Suppose also that $A(\theta)$ is a strictly concave and continuously differentiable real-valued function with the following boundary conditions to ensure interior solutions to the choice of technology: $A'(0) > 0$ and $A'(2) = -\infty$ (where A' denotes A's derivative), and $B(\mathbf{L})$ is an increasing function. $F(Z,\mathbf{L},\theta)$ satisfies all the conditions of Theorem 3 at all points $Z \in \mathcal{Z} = [0,6]$ (since F is strictly concave in θ everywhere on $\mathcal{Z} \times \Theta = [0,6] \times [0,2]$), so we have local absolute equilibrium bias at all points in $\mathcal{Z} \times \Theta$.

However, $F(Z,\mathbf{L},\theta)$ is not supermodular or nor does it satisfy increasing differences in (Z,θ), since the cross-partial between Z and θ changes sign depending on whether Z is greater than or less than 4. Consequently, it can be verified that there will not be global equilibrium bias in this example.

To illustrate this, consider $\bar{Z} = 1$ and $\bar{Z}' = 5$ as two potential supply levels of factor Z and some $\bar{\mathbf{L}} \in \mathcal{L}$. It can be verified easily that $F\left(1,\bar{\mathbf{L}},\theta\right) = 1 + 7\theta/8 + A(\theta) + B\left(\bar{\mathbf{L}}\right)$, so that $\theta(1)$ satisfies $A'(\theta(1)) = -7/8$, whereas $F\left(5,\bar{\mathbf{L}},\theta\right) = 5 + 15\theta/8 + A(\theta) + B\left(\bar{\mathbf{L}}\right)$ so that $\theta(5)$ satisfies $A'(\theta(5)) = -15/8$. The strict concavity of $A(\theta)$ implies that $\theta(5) > \theta(1)$. Moreover, $w_Z(Z,\theta) = 1 + (1 - Z/4)\theta$, so $w_Z(5,\theta(5)) = 1 - \theta(5)/4 < w_Z(5,\theta(1)) = 1 - \theta(1)/4$, contrary to the claim in Theorem 4.

So why does the global theorem not work, while the local theorem does? The answer is that the local theorem, Theorem 3, only refers to changes in technology that are absolutely biased at the corresponding factor proportions. Consequently, when we change $\bar{Z}$ locally, say from 1 to 1.1, this increases θ, which is absolutely biased towards Z around 1. But this change is biased against Z when we look at $\bar{Z} = 5$. This is the fundamental reason why applying the local theorem, Theorem 3, successively will not give a global theorem and we need the additional increasing differences (supermodularity) conditions (see also footnote 31 below for further discussion on this point).

This example can also be used to illustrate that increasing differences cannot be replaced by the weaker single-crossing property, since $F(Z, \mathbf{L}, \theta)$ may satisfy single crossing both in (Z, θ) and (θ, Z). To illustrate this, let us take $\Theta = \{\theta(1), \theta(5)\}$ and suppose that $\theta(1) = 0$ and $\theta(5) = 1$. Let us continue to take $\mathcal{Z} = [0, 6]$. First to check single crossing in (Z, θ), note that since $\theta(1) = 0$, $F(\bar{Z}', \bar{\mathbf{L}}, \theta(1)) > F(\bar{Z}, \bar{\mathbf{L}}, \theta(1))$ whenever $\bar{Z}' > \bar{Z}$. Therefore, we only have to check that $F(\bar{Z}', \bar{\mathbf{L}}, \theta(5)) > F(\bar{Z}, \bar{\mathbf{L}}, \theta(5))$ whenever $\bar{Z}' > \bar{Z}$. This immediately follows from the fact that $\theta(5) = 1$, so that for all $\bar{Z}', \bar{Z} \in \mathcal{Z} = [0, 6]$ and $\bar{Z}' > \bar{Z}$, $\bar{Z}' + \left(\bar{Z}' - (\bar{Z}')^2/8\right) > \bar{Z} + \left(\bar{Z} - \bar{Z}^2/8\right)$. To establish single crossing in (θ, Z), let us take $\theta(1) = 0$ and $\theta(5) = 1$ and also suppose that $A(1) \geq A(0)$. In that case, single crossing in (θ, Z) requires that whenever $\bar{Z}', \bar{Z} \in \mathcal{Z} = [0, 6]$ and $\bar{Z}' > \bar{Z}$, and

$$\bar{Z} + \left(\bar{Z} - \bar{Z}^2/8\right) + A(1) + B(\bar{\mathbf{L}}) > \bar{Z} + A(0) + B(\bar{\mathbf{L}})$$

it must also be the case that

$$\bar{Z}' + \left(\bar{Z}' - (\bar{Z}')^2/8\right) + A(1) + B(\bar{\mathbf{L}}) > \bar{Z}' + A(0) + B(\bar{\mathbf{L}}).$$

It is straightforward to verify that the first inequality will hold for all $\bar{Z} \in (0, 6]$ since, in this case, $\left(\bar{Z} - \bar{Z}^2/8\right) > 0 \geq A(0) - A(1)$. This implies that for all $\bar{Z}', \bar{Z} \in \mathcal{Z} = [0, 6]$ and $\bar{Z}' > \bar{Z}$, we have $\left(\bar{Z}' - (\bar{Z}')^2/8\right) > 0 \geq A(0) - A(1)$, establishing single crossing in (θ, Z). Since by Lemma 3 in Appendix B, when $\mathcal{Z}$ and Θ are chains, single crossing in (Z, θ) and (θ, Z) implies quasi-supermodularity in (Z, θ), this result also implies that increasing differences in (Z, θ) cannot be replaced with quasi-supermodularity in (Z, θ).

Finally, this example also shows that the assumption that the function needs to exhibit increasing differences in (Z, θ) on the *convex hull* of $\mathcal{Z}$, $\bar{\mathcal{Z}}$, cannot be dispensed with. In particular, if we take $\mathcal{Z} = \{1, 5\}$ and $\Theta = \{\theta(1), \theta(5)\}$, it can be verified that the function F here satisfies supermodularity on $\mathcal{Z} \times \Theta$, and hence exhibits increasing differences in (Z, θ) on $\mathcal{Z} \times \Theta$ (see Lemma 2 in Appendix B). However, it fails to satisfy supermodularity and increasing differences on $\bar{\mathcal{Z}} \times \Theta$, where $\bar{\mathcal{Z}} = [1, 5]$.[28]

There is a natural generalization of Theorem 4 in which the supplies of a set of factors change simultaneously. This is presented in the next theorem. Let the production function be

[28] To see why it is necessary for F to be supermodular or exhibit increasing differences in (Z, θ) over the convex hull of $\mathcal{Z}$, note that the supermodularity of F implies that for $Z'' > Z'$ and $\theta'' > \theta'$, we have

$$F(Z'', \mathbf{L}, \theta'') + F(Z', \mathbf{L}, \theta') \geq F(Z'', \mathbf{L}, \theta') + F(Z', \mathbf{L}, \theta'').$$

Now, assuming differentiability and applying the Fundamental Theorem of Calculus (e.g., Rudin, 1964, Theorem

$F(\mathbf{Z}, \mathbf{L}, \theta)$, where $\mathbf{Z} = (Z_1, ..., Z_N)$. Define the marginal products in the usual way as

$$w_{Zj} = \frac{\partial F(\mathbf{Z}, \mathbf{L}, \theta)}{\partial Z_j} \text{ for } j = 1, ..., N.$$

The notion of equilibrium bias generalizes naturally.

Definition 12 Let $\bar{\mathbf{Z}} \in \mathcal{Z} \subset \mathbb{R}_+^N$, $\bar{\mathbf{L}} \in \mathcal{L}$ and $\theta(\bar{\mathbf{Z}}, \bar{\mathbf{L}})$ be the equilibrium technology choice in an economy with factor supplies $(\bar{\mathbf{Z}}, \bar{\mathbf{L}})$. We say that there is *global absolute equilibrium bias* if for any $\bar{\mathbf{Z}}', \bar{\mathbf{Z}} \in \mathcal{Z}$, $\bar{\mathbf{Z}}' \geq \bar{\mathbf{Z}}$ implies

$$w_{Zj}\left(\tilde{\mathbf{Z}}, \bar{\mathbf{L}}, \theta(\bar{\mathbf{Z}}', \bar{\mathbf{L}})\right) \geq w_{Zj}\left(\tilde{\mathbf{Z}}, \bar{\mathbf{L}}, \theta(\bar{\mathbf{Z}}, \bar{\mathbf{L}})\right) \text{ for all } \left(\tilde{\mathbf{Z}}, \bar{\mathbf{L}}\right) \in \mathcal{Z} \times \mathcal{L} \text{ and for all } j = 1, ..., N.$$

Once again, this definition can be strengthened by introducing strict inequalities.

Theorem 5 *(Generalized Global Equilibrium Bias)* Consider Economy D, C or M. Suppose that $\mathcal{Z}$ and Θ are lattices, let $\bar{\mathcal{Z}}$ be the convex hull of $\mathcal{Z}$, let $\theta(\bar{\mathbf{Z}}, \bar{\mathbf{L}})$ be the equilibrium technology at factor proportions $(\bar{\mathbf{Z}}, \bar{\mathbf{L}})$, and suppose that $F(\mathbf{Z}, \mathbf{L}, \theta)$ is continuously differentiable in $\mathbf{Z}$, supermodular in θ on Θ for all $\mathbf{Z} \in \bar{\mathcal{Z}}$ and $\mathbf{L} \in \mathcal{L}$, and exhibits strictly increasing differences in $(\mathbf{Z}, \theta)$ on $\bar{\mathcal{Z}} \times \Theta$ for all $\mathbf{L} \in \mathcal{L}$, then there is *global absolute equilibrium bias*, i.e., for any $\bar{\mathbf{Z}}', \bar{\mathbf{Z}} \in \mathcal{Z}$, $\bar{\mathbf{Z}}' \geq \bar{\mathbf{Z}}$ implies

$$\theta(\bar{\mathbf{Z}}', \bar{\mathbf{L}}) \geq \theta(\bar{\mathbf{Z}}, \bar{\mathbf{L}}) \text{ for all } \bar{\mathbf{L}} \in \mathcal{L}$$

and

$$w_{Zj}\left(\tilde{\mathbf{Z}}, \bar{\mathbf{L}}, \theta(\bar{\mathbf{Z}}', \bar{\mathbf{L}})\right) \geq w_{Zj}\left(\tilde{\mathbf{Z}}, \bar{\mathbf{L}}, \theta(\bar{\mathbf{Z}}, \bar{\mathbf{L}})\right) \text{ for all } \left(\tilde{\mathbf{Z}}, \bar{\mathbf{L}}\right) \in \mathcal{Z} \times \mathcal{L} \text{ and for all } j = 1, ..., N.$$

Proof. The proof is analogous to that of Theorem 4 and follows from Theorem 10 in Appendix B given the supermodularity of $F(\mathbf{Z}, \mathbf{L}, \theta)$ in θ and strict increasing differences in $(\mathbf{Z}, \theta)$ on $\bar{\mathcal{Z}} \times \Theta$. ■

It is clear that corollaries to this theorem similar to those to Theorem 4 can be stated with slightly stronger conditions. I omit these to avoid repetition. Also, as in Theorem 4,

6.16) twice and using the definition of w_Z, we have

$$\int_{Z'}^{Z''} \int_{\theta'}^{\theta''} \frac{\partial w_Z(Z, \mathbf{L}, \theta)}{\partial \theta} d\theta dZ \geq 0.$$

However, this does not guarantee that

$$\int_{\theta'}^{\theta''} \frac{\partial w_Z(Z, \mathbf{L}, \theta)}{\partial \theta} d\theta \geq 0$$

for all $Z \in [Z', Z'']$ unless F is supermodular over the convex hull of $\{Z', Z''\}$.

supermodularity in θ can be weakened to quasi-supermodularity in θ, or strict increasing differences can be relaxed to increasing differences and the comparison of $\theta\left(\bar{\mathbf{Z}}', \bar{\mathbf{L}}\right)$ to $\theta\left(\bar{\mathbf{Z}}, \bar{\mathbf{L}}\right)$ would apply for the greatest and the least elements of the equilibrium technology set.

5 Strong Absolute Equilibrium Bias

The results in Section 4 concern what was referred to as "weak" bias in the sense that they compare marginal products at a given level of factor supplies (in response to a change in θ induced by a change in Z). This section provides the conditions under which equilibrium bias will be "strong" in the sense that once technology has adjusted, the increase in the supply of factor Z will increase its marginal product (price). As noted in the Introduction, this is particularly important because it emphasizes the central role of the equilibrium structure in the analysis here, since such a result would not be possible in the neoclassical production theory.

Example 1 above illustrated the possibility of strong relative bias where technology might be so responsive to factor supply changes that when a factor becomes more abundant, its relative price and marginal product increase rather than decrease. Although somewhat counterintuitive at first, this is also a possibility in the class of models studied here. But we will see that it requires some type of non-convexity either in the technology set, Θ, or in the production possibilities set by allowing for a structure similar to that of Economy C or Economy M. First, I define strong absolute bias, and to simplify the discussion, from now on, I focus on changes in a single factor. Recall throughout that equilibrium technology is still a solution to the maximization problem in (33).

Definition 13 Suppose that $N = 1$. Let $\theta(Z, \mathbf{L}) \in \Theta$ be the equilibrium technology choice in an economy with factor proportions $(Z, \mathbf{L})$. We say that there is *strong absolute equilibrium bias* at $\left(\{\bar{Z}, \bar{Z}'\}, \bar{\mathbf{L}}\right)$ if for some $\bar{\mathbf{L}} \in \mathcal{L}$ and $\bar{Z}, \bar{Z}' \in \mathcal{Z}$ with $\bar{Z} > \bar{Z}'$, we have

$$w_Z\left(\bar{Z}', \bar{\mathbf{L}}, \theta\left(\bar{Z}', \bar{\mathbf{L}}\right)\right) > w_Z\left(\bar{Z}, \bar{\mathbf{L}}, \theta\left(\bar{Z}, \bar{\mathbf{L}}\right)\right).$$

Similarly, suppose that $\Theta \subset \mathbb{R}^K$, $w_Z\left(\bar{Z}, \bar{\mathbf{L}}, \theta\left(\bar{Z}, \bar{\mathbf{L}}\right)\right)$ is differentiable in Z and $\partial\theta_j\left(\bar{Z}, \bar{\mathbf{L}}\right)/\partial Z$ exists at $\left(\bar{Z}, \bar{\mathbf{L}}\right)$ for all $j = 1, ..., K$. Then we say that there is *strong absolute equilibrium bias* at $\left(\bar{Z}, \bar{\mathbf{L}}\right) \in \mathcal{Z} \times \mathcal{L}$ if

$$\frac{dw_Z\left(\bar{Z}, \bar{\mathbf{L}}, \theta\left(\bar{Z}, \bar{\mathbf{L}}\right)\right)}{dZ} = \frac{\partial w_Z\left(\bar{Z}, \bar{\mathbf{L}}, \theta\left(\bar{Z}, \bar{\mathbf{L}}\right)\right)}{\partial Z} + \sum_{j=1}^{K} \frac{\partial w_Z\left(\bar{Z}, \bar{\mathbf{L}}, \theta\left(\bar{Z}, \bar{\mathbf{L}}\right)\right)}{\partial \theta_j} \frac{\partial \theta_j\left(\bar{Z}, \bar{L}\right)}{\partial Z} > 0.$$

Note that in this definition I use $dw_Z\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)/dZ$ to denote the total derivative, while $\partial w_Z\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)/\partial Z$ denotes the partial derivative holding $\theta=\theta\left(\bar{Z},\bar{\mathbf{L}}\right)$. The next theorem shows that there cannot be strong absolute bias in Economy D if Θ is a convex subset of $\mathbb{R}^K$.

Theorem 6 *(No Strong Bias in Economy D)* Suppose that Θ is a convex subset of $\mathbb{R}^K$, F is twice continuously differentiable in (Z,θ), let the equilibrium technology at factor supplies $\left(\bar{Z},\bar{\mathbf{L}}\right)$ be $\theta\left(\bar{Z},\bar{\mathbf{L}}\right)$, and assume that $\partial\theta_j\left(\bar{Z},\bar{\mathbf{L}}\right)/\partial Z$ exists at $\left(\bar{Z},\bar{\mathbf{L}}\right)$ for all $j=1,...,K$. Then there cannot be *strong absolute bias* in Economy D.

Proof. Let us start with the local result and the case with $\theta\in\mathbb{R}$. Let factor supplies be $\left(\bar{Z},\bar{\mathbf{L}}\right)$. Strong absolute bias corresponds to

$$\frac{dw_Z\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)}{dZ}=\frac{\partial w_Z\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)}{\partial Z}+\frac{\partial w_Z\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)}{\partial\theta}\frac{\partial\theta\left(\bar{Z},\bar{L}\right)}{\partial Z}>0.$$

This is equivalent to

$$\frac{dw_Z\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)}{dZ}=\frac{\partial^2 F\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)}{\partial Z^2}+\frac{\partial w_Z\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)}{\partial\theta}\frac{\partial\theta\left(\bar{Z},\bar{L}\right)}{\partial Z}>0.$$

Recall from the proof of Theorem 3 that when $\theta\left(\bar{Z},\bar{\mathbf{L}}\right)$ is in the interior of Θ, the first-order condition (37) holds, and we have:

$$\frac{\partial\theta\left(\bar{Z},\bar{\mathbf{L}}\right)}{\partial Z}=-\frac{\partial^2 F\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)/\partial\theta\partial Z}{\partial^2 F\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)/\partial\theta^2}=-\frac{\partial w_Z\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)/\partial\theta}{\partial^2 F\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)/\partial\theta^2}.$$

so strong absolute bias would imply

$$\frac{dw_Z\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)}{dZ}=\frac{\partial^2 F\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)}{\partial Z^2}-\frac{\left(\partial^2 F\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)/\partial\theta\partial Z\right)^2}{\partial^2 F\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)/\partial\theta^2}>0. \tag{41}$$

To see that this is impossible, first note that since $\partial\theta\left(\bar{Z},\bar{L}\right)/\partial Z$ exists, $\partial^2 F\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)/\partial\theta^2<0$ (from the non-singularity, $\partial^2 F\left(\bar{Z},\bar{\mathbf{L}},\theta\right)/\partial\theta_j^2\neq 0$ combined with the fact that $\theta\left(\bar{Z},\bar{L}\right)$ is a solution to (33), so that $\partial^2 F\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)/\partial\theta^2\leq 0$); and second that the joint concavity of $F\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)$ in (Z,θ) implies that the Hessian of F in (Z,θ), $\nabla^2_{(Z,\theta)(Z,\theta)}F$, is negative semi-definite, thus every principle minor of $\nabla^2_{(Z,\theta)(Z,\theta)}F$ of even order has to be nonnegative (see, e.g., Simon and Blume, 1994, Theorem 16.2). This implies

$$\partial^2 F\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)/\partial\theta^2\times\left(\partial^2 F\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)/\partial Z^2\right)\geq\left(\partial^2 F\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)/\partial\theta\partial Z\right)^2,$$

which combined with $\partial^2 F\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)/\partial\theta^2<0$ contradicts (41), proving the claim for the case of $\theta\in\mathbb{R}$ and in the interior of Θ. When θ is at the boundary of Θ, either (37) holds, in which case the same argument applies (since by hypothesis $\partial \theta (\bar{Z}, \bar{L}) / \partial Z$ exists even at this point). Alternatively, $\partial F (\bar{Z}, \bar{\mathbf{L}}, \theta (\bar{Z}, \bar{\mathbf{L}})) / \partial \theta < 0$. However, in this case since F is twice continuously differentiable in (Z, θ) and $\partial \theta (\bar{Z}, \bar{L}) / \partial Z$ exists, a sufficiently small change in Z will leave $\partial F (\bar{Z}, \bar{\mathbf{L}}, \theta (\bar{Z}, \bar{\mathbf{L}})) / \partial \theta < 0$ and thus $\partial \theta (\bar{Z}, \bar{\mathbf{L}}) / \partial Z = 0$. Consequently, $dw_Z (\bar{Z}, \bar{\mathbf{L}}, \theta (\bar{Z}, \bar{\mathbf{L}})) / dZ = \partial w_Z (\bar{Z}, \bar{\mathbf{L}}, \theta (\bar{Z}, \bar{\mathbf{L}})) / \partial Z \leq 0$.

Next, to prove this result with multiple dimensions of technology, i.e., with $\theta \in \mathbb{R}^K$ for $K > 1$, note that when $\theta (\bar{Z}, \bar{\mathbf{L}})$ is in the interior of Θ, we have

$$\begin{aligned} \frac{dw_Z (\bar{Z}, \bar{\mathbf{L}}, \theta (\bar{Z}, \bar{\mathbf{L}}))}{dZ} &= \frac{\partial^2 F (\bar{Z}, \bar{\mathbf{L}}, \theta (\bar{Z}, \bar{\mathbf{L}}))}{\partial Z^2} \\ &- [\nabla^2_{\theta Z} F (\bar{Z}, \bar{\mathbf{L}}, \theta (\bar{Z}, \bar{\mathbf{L}}))]' [\nabla^2_{\theta\theta} F (\bar{Z}, \bar{\mathbf{L}}, \theta (\bar{Z}, \bar{\mathbf{L}}))]^{-1} [\nabla^2_{\theta Z} F (\bar{Z}, \bar{\mathbf{L}}, \theta (\bar{Z}, \bar{\mathbf{L}}))]. \end{aligned} \tag{42}$$

Since $\theta (\bar{Z}, \bar{\mathbf{L}})$ is a solution to (33), $\nabla^2_{\theta\theta} F (\bar{Z}, \bar{\mathbf{L}}, \theta (\bar{Z}, \bar{\mathbf{L}}))$ is negative semi-definite and symmetric (since it is a Hessian). Moreover, since $\nabla_Z \theta (\bar{Z}, \bar{\mathbf{L}})$ exists by hypothesis, $\nabla^2_{\theta\theta} F (\bar{Z}, \bar{\mathbf{L}}, \theta (\bar{Z}, \bar{\mathbf{L}}))$ is non-singular, so it is negative definite and symmetric. This implies that its inverse $[\nabla^2_{\theta\theta} F (\bar{Z}, \bar{\mathbf{L}}, \theta (\bar{Z}, \bar{\mathbf{L}}))]^{-1}$ is also negative definite and symmetric, and moreover $\partial^2 F (\bar{Z}, \bar{\mathbf{L}}, \theta) / \partial Z^2 \leq 0$ by the concavity of F in Z (from Assumption 1, 1', 2 or 2').

Lemma 4 in Appendix B shows that an $n \times n$ matrix

$$B = \begin{pmatrix} C & v \\ v' & b \end{pmatrix},$$

where C is a $(n-1) \times (n-1)$ symmetric negative definite, b is a scalar, and v is a $(n-1) \times 1$ column vector, is negative semi-definite *if and only if* $b - v' C^{-1} v \leq 0$ where C^{-1} is the inverse of C. Let us now apply this lemma with $b = \partial^2 F (\bar{Z}, \bar{\mathbf{L}}, \theta) / \partial Z^2 \leq 0$, $C = [\nabla^2_{\theta\theta} F (\bar{Z}, \bar{\mathbf{L}}, \theta (\bar{Z}, \bar{\mathbf{L}}))]$, and $v = [\nabla^2_{\theta Z} F (\bar{Z}, \bar{\mathbf{L}}, \theta (\bar{Z}, \bar{\mathbf{L}}))]$, which implies that the expression in (42) is equal to $b - v' C^{-1} v$. The fact that F is jointly concave in (Z, θ) implies that the Hessian of F with respect to (Z, θ), $\nabla^2_{(Z,\theta)(Z,\theta)} F$ is negative semi-definite. Therefore, from Lemma 4, $b - v' C^{-1} v \leq 0$ and (42) cannot be positive, completing the proof of the local result. The proof for the case where $\theta (\bar{Z}, \bar{\mathbf{L}})$ is at the boundary of Θ is analogous to the one above for $\Theta \subset \mathbb{R}$.

Finally, to prove the global result, i.e., that strong bias is impossible in this economy for any change in factor supplies, note that from the Fundamental Theorem of Calculus, for any $\bar{Z}' > \bar{Z}$, we have

$$w_Z (\bar{Z}', \bar{\mathbf{L}}, \theta (\bar{Z}', \bar{\mathbf{L}})) - w_Z (\bar{Z}, \bar{\mathbf{L}}, \theta (\bar{Z}, \bar{\mathbf{L}})) = \int_{\bar{Z}}^{\bar{Z}'} \frac{dw_Z (Z, \bar{\mathbf{L}}, \theta (Z, \bar{\mathbf{L}}))}{dZ} dZ.$$

Since $dw_Z (Z, \bar{\mathbf{L}}, \theta (\bar{Z}, \bar{\mathbf{L}})) / dZ \leq 0$ for all $Z \in [\bar{Z}, \bar{Z}']$, the integral is nonpositive, establishing the global result. ■

The result in this theorem is not surprising. In Economy D, the production possibilities set is convex, so the marginal product of each factor is decreasing in its supply even after technology adjusts. In contrast, once we allow for non-convexities (and factor demands and technology to be chosen by different agents), the results are very different. To illustrate the importance of non-convexities, I now look at Economy D with a non-convex technology set Θ,[29] and at Economies C or M, which allow for natural non-convexities. I establish that in both cases strong absolute bias is possible.

Theorem 7 *(Strong Absolute Bias) Strong absolute equilibrium bias* is possible either in Economy D with a non-convex technology set, Θ, or in Economy C or M.

This theorem will be proved by providing two examples with strong absolute equilibrium bias.

Example 4 *(Strong Absolute Bias in Economy D)* Take Economy D and suppose that $F(Z, \mathbf{L}, \theta) = Z^{1/2}\theta^{1/2} - \theta + B(\mathbf{L})$ and $\Theta = \{1, 4\}$. Imagine an increase in $\bar{Z}$ from 4 to $9 + \varepsilon$ where $\varepsilon > 0$. It is straightforward to check that for any $\bar{\mathbf{L}} \in \mathcal{L}$, $F\left(4, \bar{\mathbf{L}}, 1\right) = 2 - 1 + B\left(\bar{\mathbf{L}}\right) > F\left(4, \bar{\mathbf{L}}, 4\right) = 4 - 4 + B\left(\bar{\mathbf{L}}\right)$, so $\theta(4) = 1$. In contrast, $F\left(9 + \varepsilon, \bar{\mathbf{L}}, 4\right) = (9 + \varepsilon)^{1/2} 2 - 4 + B\left(\bar{\mathbf{L}}\right) > F\left(9 + \varepsilon, \bar{\mathbf{L}}, 1\right) = (9 + \varepsilon)^{1/2} - 1 + B\left(\bar{\mathbf{L}}\right)$, so that $\theta(9 + \varepsilon) = 4$ (in particular, the two sides are equal when $\varepsilon = 0$, and the left-hand side increases faster in ε). Therefore, an increase in $\bar{Z}$ from 4 to $9 + \varepsilon$ will induce a change in technology from $\theta(4) = 1$ to $\theta(9 + \varepsilon) = 4$. The price (marginal product) of factor Z is given by $w_Z\left(\bar{Z}, \bar{\mathbf{L}}, \theta\right) = \left(\theta/\bar{Z}\right)^{1/2}/2$, so the change in this price resulting from the increase in $\bar{Z}$ (after technology adjusts) is $w_Z\left(\bar{Z} = 9 + \varepsilon, \bar{\mathbf{L}}, 4\right) - w_Z\left(\bar{Z} = 4, \bar{\mathbf{L}}, 1\right) = (4/(9 + \varepsilon))^{1/2}/2 - (1/4)^{1/2}/2 \simeq 1/3 - 1/4 > 0$ for ε sufficiently small, establishing the possibility of strong absolute bias in Economy D with a non-convex technology set.

Example 5 *(Strong Absolute Bias in Economy C or M)* Next, consider Economy C or M, and to illustrate that a non-convex technology set is not necessary in these economies, take $\Theta = \mathbb{R}$. Suppose $F(Z, \mathbf{L}, \theta) = 4Z^{1/2} + Z\theta - \theta^2/2 + B(\mathbf{L})$, which is not jointly concave in Z and θ (for $Z > 1$) but is strictly concave in Z and θ individually. As Theorem 8 below will show, this is a crucial feature in generating strong (absolute) bias. Now consider a change from $\bar{Z} = 1$ to $\bar{Z} = 4$. Clearly, the first-order necessary and sufficient condition for technology choice gives

[29] In Economy D, when Assumption 1 applies the technology set Θ is also assumed to be convex. This assumption can be relaxed. Recall also that convexity of Θ is not required by Assumption 1', which only requires $\mathcal{L}$ and $\mathcal{Z}$ to be convex.

$\theta\left(\bar{Z},\bar{\mathbf{L}}\right)=\theta\left(\bar{Z}\right)=\bar{Z}$. Therefore, $\theta\left(\bar{Z}=1\right)=1$ while $\theta\left(\bar{Z}=4\right)=4$. Moreover, for any $\bar{\mathbf{L}}\in\mathcal{L}$, $w_Z\left(\bar{Z},\bar{\mathbf{L}},\theta\right)=2Z^{-1/2}+\theta$. Therefore, $w_Z\left(\bar{Z}=1,\bar{\mathbf{L}},\theta\left(1\right)\right)=3<w_Z\left(\bar{Z}=4,\bar{\mathbf{L}},\theta\left(4\right)\right)=5$, establishing strong (absolute) equilibrium bias between $\bar{Z}=1$ to $\bar{Z}=4$. In fact, Theorem 8 implies that there will be local strong equilibrium bias in this example for all $\bar{Z}\geq 1$, and Theorem 9 then implies that there will be global strong equilibrium bias between any $\bar{Z}'$ and $\bar{Z}$ with $\bar{Z}'>\bar{Z}\geq 1$.

The importance of Theorem 7 is that, contrary to the predictions of the standard production theory, where the increase in the supply of a factor always reduces its price (and marginal product), with endogenous technology choice or technological change, the price of a factor which has become more abundant can increase. Examples 4 and 5 show that it is straightforward to construct economies in which there is such strong bias.

This theorem also distinguishes the approach in this paper from the literature on the LeChatelier principle, which looks at the decision problem of a single firm. As is well-known, the firm's demand curve for a factor is always downward sloping in its own price (e.g., Mas-Colell, Winston and Green, 1995, Proposition 5.C.2), so the equilibrium structure (in particular, the equilibrium with aggregate non-convexities) is important for the results in this paper, especially for the possibility of strong equilibrium bias.

The fact that strong equilibrium bias is possible in Economy D when the technology set Θ is non-convex is also interesting. Although many existing approaches to technology, such as the models of endogenous technological progress (e.g., Romer, 1990, Grossman and Helpman, 1991, Aghion and Howitt, 1992), view technology as a scalar in a convex set, as already discussed in the Introduction, for many important technological choices, switching between discrete technologies may be quite important. If this is the case, allowing Θ to be non-convex is important and realistic (recall that Theorems 4 and 5 above do not require Θ to be convex).

Finally, as stated in the Introduction and already hinted in the discussion, "greater non-convexity" makes it more likely that the economy will feature strong absolute bias. This is formalized in the next theorem. Recall that in Economy C or M, $F\left(Z,\mathbf{L},\theta\right)=G\left(Z,\mathbf{L},\theta\right)-C\left(\theta\right)$, so marginal product of Z is equivalently given by the derivative of function F or G. Recall also that $F\left(Z,\mathbf{L},\theta\right)$ is always concave in $\left(Z,\mathbf{L}\right)$ (from Assumption 1, 1', 2 or 2') and has to be locally concave in θ for $\theta\left(\bar{Z},\bar{\mathbf{L}}\right)$ to be an equilibrium technology (i.e., a solution to the maximization problem in (33)). Recall that if F is jointly concave in $\left(Z,\theta\right)$ at $\left(Z,\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)$, its Hessian with respect to $\left(Z,\theta\right)$, $\nabla^2F_{(Z,\theta)(Z,\theta)}$, is negative semi-definite at this point (though negative semi-definiteness is not sufficient for local joint concavity).

Theorem 8 *(Non-Convexity and Strong Bias)* Consider Economy C or M. Suppose that Θ is a convex subset of $\mathbb{R}^K$, F is twice continuously differentiable in (Z, θ), let $\theta(\bar{Z}, \bar{\mathbf{L}})$ be the equilibrium technology at factor supplies $(\bar{Z}, \bar{\mathbf{L}})$ and assume that $\theta(\bar{Z}, \bar{\mathbf{L}})$ is in the interior of Θ and that $\partial\theta_j(\bar{Z}, \bar{\mathbf{L}})/\partial Z$ exists at $(\bar{Z}, \bar{\mathbf{L}})$ for all $j = 1, ..., K$. Then there is strong absolute bias at $(\bar{Z}, \bar{\mathbf{L}})$ *if and only if* $F(Z, \mathbf{L}, \theta)$'s Hessian in (Z, θ), $\nabla^2 F_{(Z,\theta)(Z,\theta)}$, *is not negative semi-definite* at $(\bar{Z}, \bar{\mathbf{L}}, \theta(\bar{Z}, \bar{\mathbf{L}}))$.

Proof. Let us start with the case where $\Theta \subset \mathbb{R}$. Since, by hypothesis, θ is in the interior of Θ, the first-order condition, equation (37) from the proof of Theorem 3, holds. Then recall the proof of Theorem 6 and in particular, equation (41), where it was established that for the case of $\theta \in \mathbb{R}$:

$$\frac{dw_Z(\bar{Z}, \bar{\mathbf{L}}, \theta(\bar{Z}, \bar{\mathbf{L}}))}{dZ} = \frac{\partial^2 F(\bar{Z}, \bar{\mathbf{L}}, \theta(\bar{Z}, \bar{\mathbf{L}}))}{\partial Z^2} - \frac{\left(\partial^2 F(\bar{Z}, \bar{\mathbf{L}}, \theta(\bar{Z}, \bar{\mathbf{L}}))/\partial\theta\partial Z\right)^2}{\partial^2 F(\bar{Z}, \bar{\mathbf{L}}, \theta(\bar{Z}, \bar{\mathbf{L}}))/\partial\theta^2}.$$

Again from Assumption 1, 1', 2 or 2', F is concave in Z, so $\partial^2 F(\bar{Z}, \bar{\mathbf{L}}, \theta)/\partial Z^2 \leq 0$, and from the fact that $\theta(\bar{Z}, \bar{\mathbf{L}})$ is a solution to (33) and $\partial\theta(\bar{Z}, \bar{\mathbf{L}})/\partial Z$ exists, we also have $\partial^2 F(\bar{Z}, \bar{\mathbf{L}}, \theta(\bar{Z}, \bar{\mathbf{L}}))/\partial\theta^2 < 0$. Then the fact that $F(\bar{Z}, \bar{\mathbf{L}}, \theta(\bar{Z}, \bar{\mathbf{L}}))$'s Hessian, $\nabla^2 F_{(Z,\theta)(Z,\theta)}$, is not negative semi-definite at $(\bar{Z}, \bar{\mathbf{L}}, \theta(\bar{Z}, \bar{\mathbf{L}}))$ implies that

$$\left(\partial^2 F(\bar{Z}, \bar{\mathbf{L}}, \theta(\bar{Z}, \bar{\mathbf{L}}))/\partial\theta^2\right) \times \left(\partial^2 F(\bar{Z}, \bar{\mathbf{L}}, \theta(\bar{Z}, \bar{\mathbf{L}}))/\partial Z^2\right) < \left(\partial^2 F(\bar{Z}, \bar{\mathbf{L}}, \theta(\bar{Z}, \bar{\mathbf{L}}))/\partial\theta\partial Z\right)^2.$$

Since at the optimal technology choice, $\partial^2 F(\bar{Z}, \bar{\mathbf{L}}, \theta(\bar{Z}, \bar{\mathbf{L}}))/\partial\theta^2 < 0$, this immediately yields

$$\frac{dw_Z(\bar{Z}, \bar{\mathbf{L}}, \theta(\bar{Z}, \bar{\mathbf{L}}))}{dZ} > 0,$$

establishing strong absolute bias at $(\bar{Z}, \bar{\mathbf{L}}, \theta(\bar{Z}, \bar{\mathbf{L}}))$ as claimed in the theorem.

Conversely, if $\nabla^2 F_{(Z,\theta)(Z,\theta)}$ is negative semi-definite at $(\bar{Z}, \bar{\mathbf{L}}, \theta(\bar{Z}, \bar{\mathbf{L}}))$, then

$$\left(\partial^2 F(\bar{Z}, \bar{\mathbf{L}}, \theta(\bar{Z}, \bar{\mathbf{L}}))/\partial\theta^2\right) \times \left(\partial^2 F(\bar{Z}, \bar{\mathbf{L}}, \theta(\bar{Z}, \bar{\mathbf{L}}))/\partial Z^2\right) \geq \left(\partial^2 F(\bar{Z}, \bar{\mathbf{L}}, \theta(\bar{Z}, \bar{\mathbf{L}}))/\partial\theta\partial Z\right)^2,$$

which, together with $\partial^2 F(\bar{Z}, \bar{\mathbf{L}}, \theta(\bar{Z}, \bar{\mathbf{L}}))/\partial\theta^2 < 0$, implies that $dw_Z(\bar{Z}, \bar{\mathbf{L}}, \theta(\bar{Z}, \bar{\mathbf{L}}))/dZ \leq 0$, establishing that for strong bias at $(\bar{Z}, \bar{\mathbf{L}}, \theta(\bar{Z}, \bar{\mathbf{L}}))$ we need $\nabla^2 F_{(Z,\theta)(Z,\theta)}$ not to be negative semi-definite at $(\bar{Z}, \bar{\mathbf{L}}, \theta(\bar{Z}, \bar{\mathbf{L}}))$.

Now for the general case where $\Theta \subset \mathbb{R}^K$ and $\theta(\bar{Z}, \bar{\mathbf{L}})$ is in the interior of Θ, the overall change in the price of factor Z is given by (42) in the proof of Theorem 6, i.e.,

$$\begin{aligned}\frac{dw_Z(\bar{Z}, \bar{\mathbf{L}}, \theta(\bar{Z}, \bar{\mathbf{L}}))}{dZ} &= \frac{\partial^2 F(\bar{Z}, \bar{\mathbf{L}}, \theta(\bar{Z}, \bar{\mathbf{L}}))}{\partial Z^2} \\ &- \left[\nabla^2_{\theta Z} F(\bar{Z}, \bar{\mathbf{L}}, \theta(\bar{Z}, \bar{\mathbf{L}}))\right]' \left[\nabla^2_{\theta\theta} F(\bar{Z}, \bar{\mathbf{L}}, \theta(\bar{Z}, \bar{\mathbf{L}}))\right]^{-1} \left[\nabla^2_{\theta Z} F(\bar{Z}, \bar{\mathbf{L}}, \theta(\bar{Z}, \bar{\mathbf{L}}))\right].\end{aligned} \tag{43}$$

Again by the same arguments, $\partial^2 F\left(\bar{Z},\bar{\mathbf{L}},\theta\right)/\partial Z^2 \leq 0$ and $\nabla^2_{\theta\theta} F\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)$ is negative definite and symmetric (which implies that its inverse $\left[\nabla^2_{\theta\theta} F\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)\right]^{-1}$ is also negative definite and symmetric). Suppose that $\nabla^2 F_{(Z,\theta)(Z,\theta)}$ is not negative semi-definite at $\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)$. Then from Lemma 4 in Appendix B and using the same notation as in the proof of Theorem 6, let $B = \nabla^2 F_{(Z,\theta)(Z,\theta)}$, $b = \partial^2 F\left(\bar{Z},\bar{\mathbf{L}},\theta\right)/\partial Z^2 \leq 0$, $C = \left[\nabla^2_{\theta\theta} F\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)\right]$, and $v = \left[\nabla^2_{\theta Z} F\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)\right]$, so that (43) is equal to $b - v'C^{-1}v$ evaluated at $\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)$. Lemma 4 immediately implies that if $\nabla^2 F_{(Z,\theta)(Z,\theta)}$ is not negative semi-definite at $\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)$, then $b - v'C^{-1}v > 0$, so that $dw_Z\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)/dZ > 0$ and there is strong bias at $\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)$.

Conversely, again from Lemma 4, if $\nabla^2 F_{(Z,\theta)(Z,\theta)}$ is negative semi-definite at $\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)$, then $b - v'C^{-1}v \leq 0$ and $dw_Z\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)/dZ \leq 0$, so that there is no strong bias at $\left(\bar{Z},\bar{\mathbf{L}},\theta\left(\bar{Z},\bar{\mathbf{L}}\right)\right)$, completing the proof. ■

This theorem therefore shows that in Economy C or M strong absolute bias will obtain *if and only if* the Hessian of the function $F(Z,\mathbf{L},\theta)$ fails to negative semi-definite, which loosely corresponds to F failing to be jointly concave in (Z,θ). It therefore highlights the importance of non-convexities in generating strong equilibrium bias of technology.[30]

More specifically, recall that for Economies C and M, we have Z and θ chosen by different agents. For example, in Economy M, final good producers choose their input demands, while the technology monopolist chooses technology. This implies that we are at the maximum of F when we change only Z or only θ. But this does not guarantee that we are at the maximum in the entire (Z,θ) plane. In other words, the equilibrium may be a *saddle point* rather than a maximum of the function F. When this is the case, a change in Z will induce θ to change in the direction of further increasing F, and consequently, the marginal product of factor Z will increase. Contrasting this result with Theorem 6, we see the importance of the equilibrium structure and non-convexity. As shown in Theorem 6, in Economy D with a convex technology set Θ, equilibrium ensures that we are at a maximum, so strong equilibrium bias is not possible. Strong equilibrium bias is only possible when equilibrium results from the interaction of choices by different agents (e.g., final good producers and the technology monopolist), or when the technology set Θ is itself non-convex.

Note also that Theorem 8 not only specifies the conditions for strong equilibrium bias, but also highlights that these are not very restrictive. In fact, inspection of Example 5 shows

[30] The assumption that $\theta\left(\bar{Z},\bar{\mathbf{L}}\right)$ is in the interior of Θ is adopted to obtain an "if and only if" theorem. When $\theta\left(\bar{Z},\bar{\mathbf{L}}\right)$ is at the boundary of Θ, strong equilibrium bias is again possible, but failure of negative semi-definiteness is no longer sufficient.

that it is very straightforward to construct cases in which equilibria in Economies C and M correspond to saddle points, and thus satisfy the conditions of Theorem 8.

Finally, it is also possible to provide a generalization of Theorem 8 for large changes in supplies (corresponding to strong bias between factor supplies $\{\bar{Z}, \bar{Z}'\}$ as in the first part of Definition 13). In particular, we have:[31]

Theorem 9 ***(Non-Convexity and Global Strong Bias)*** Consider Economy C or M. Suppose that Θ is a convex subset of $\mathbb{R}^K$, F is twice continuously differentiable in (Z, θ), let $\bar{Z}, \bar{Z}' \in \mathcal{Z}$, with $\bar{Z}' > \bar{Z}$, $\bar{\mathbf{L}} \in \mathcal{L}$, and let $\theta\left(\tilde{Z}, \bar{\mathbf{L}}\right)$ be the equilibrium technology at factor supplies $\left(\tilde{Z}, \bar{\mathbf{L}}\right)$ and assume that $\theta\left(\tilde{Z}, \bar{\mathbf{L}}\right)$ is in the interior of Θ and that $\partial\theta_j\left(\tilde{Z}, \bar{\mathbf{L}}\right)/\partial Z$ exists at $\left(\tilde{Z}, \bar{\mathbf{L}}\right)$ for all $j = 1, ..., K$ for all $\tilde{Z} \in \left[\bar{Z}, \bar{Z}'\right]$. Then there is *strong absolute bias* at $\left(\left\{\bar{Z}, \bar{Z}'\right\}, \bar{\mathbf{L}}\right)$ if $F(Z, \mathbf{L}, \theta)$'s Hessian, $\nabla^2 F_{(Z,\theta)(Z,\theta)}$, fails to be negative semi-definite at $\left(\tilde{Z}, \bar{\mathbf{L}}, \theta\left(\tilde{Z}, \bar{\mathbf{L}}\right)\right)$ for all $\tilde{Z} \in \left[\bar{Z}, \bar{Z}'\right]$.

Proof. The proof follows from the Fundamental Theorem of Calculus and the proof of Theorem 8. Take $\bar{Z}$ and $\bar{Z}' > \bar{Z}$ in $\mathcal{Z}$ and fix $\bar{\mathbf{L}} \in \mathcal{L}$. Then

$$w_Z\left(\bar{Z}', \bar{\mathbf{L}}, \theta\left(\bar{Z}', \bar{\mathbf{L}}\right)\right) - w_Z\left(\bar{Z}, \bar{\mathbf{L}}, \theta\left(\bar{Z}, \bar{\mathbf{L}}\right)\right) = \int_{\bar{Z}}^{\bar{Z}'} \frac{dw_Z\left(Z, \bar{\mathbf{L}}, \theta\left(Z, \bar{\mathbf{L}}\right)\right)}{dZ} dZ. \tag{44}$$

The hypotheses of the theorem, combined with the proof of Theorem 8, imply that $dw_Z\left(Z, \bar{\mathbf{L}}, \theta\left(Z, \bar{\mathbf{L}}\right)\right)/dZ > 0$ for all $Z \in \left[\bar{Z}, \bar{Z}'\right]$, so (44) is positive, establishing the result. ∎

The conditions of Theorem 9 are more demanding than Theorem 8, since they require that the Hessian of F with respect to (Z, θ) should fail to be negative semi-definite at all points $\tilde{Z} \in \left[\bar{Z}, \bar{Z}'\right]$. Moreover, this theorem is weaker than Theorem 8, since it states that failure of negative semi-definiteness of the Hessian of F between $\bar{Z}$ and $\bar{Z}'$ is sufficient to ensure strong absolute bias between $\bar{Z}$ and $\bar{Z}'$, but does not state that it is necessary (and it is straightforward to check that it is not). This motivated my focus on Theorem 8 for most of the discussion.

[31] At this point, we can also return to a further discussion of why the local weak bias result did not translate into a global weak bias result (without imposing further conditions), whereas the strong bias result does (recall the discussion in Example 3). In particular, one might have conjectured that an argument using the Fundamental Theorem of Calculus similar to that in the proof of Theorem 9, in particular, equation (44), may work for weak bias as well. To illustrate why this is not the case, let us suppose that $\Theta \subset \mathbb{R}$. Then:

$$\frac{dw_Z\left(\bar{Z}, \bar{\mathbf{L}}, \theta\left(\bar{Z}, \bar{\mathbf{L}}\right)\right)}{dZ} = \frac{\partial w_Z\left(\bar{Z}, \bar{\mathbf{L}}, \theta\left(\bar{Z}, \bar{\mathbf{L}}\right)\right)}{\partial Z} + \frac{\partial w_Z\left(\bar{Z}, \bar{\mathbf{L}}, \theta\left(\bar{Z}, \bar{\mathbf{L}}\right)\right)}{\partial \theta} \frac{\partial \theta}{\partial Z}.$$

Equation (44) and Theorem 9 apply to this entire term, while weak bias concerns the second part of this term. It is not possible to apply the Fundamental Theorem of Calculus just to this term, and intuitively, this is the notion discussed in Example 3, whereby an induced change in θ at some $\bar{Z}$ that is biased towards Z may be biased against Z at some different factor supply, $\bar{Z}'$.

6 Conclusion

An investigation of the determinants of equilibrium (endogenous) bias is important both for a better understanding of nature of technology adoption and technological change, and to study the distributional implications of new technologies.

In this paper, I analyzed the implications of changes in factor supplies on relative and absolute bias of technology. First, I generalized a number of existing results in the literature about relative bias in two-factor economies. These results are about how the relative marginal product of a factor responds to technological progress or technology adoption induced by changes in factor supplies. In particular, I established that with factor-augmenting technologies, equilibrium technology will always be relatively biased towards the factor that has become more abundant. Moreover, somewhat paradoxically, this induced-bias can be strong enough to increase the relative price of the factor that becomes more abundant. These results are interesting both because they apply to many macro models of economic growth, and also because they are essentially the opposite of the presumption in the literature following Hicks' (1932) seminal work.

Nevertheless, this result about relative bias does *not* generalize once we depart from a world with only factor-augmenting technologies. The reason is that with changes in technology directly affecting the elasticity of substitution between factors, the marginal product of the factor that is becoming less abundant might increase even more. This suggests that more general theorems may apply to absolute rather than relative bias.

The second part of the paper shows that this is indeed the case and provides general theorems about absolute bias, i.e., how the marginal product of a factor (rather than its relative marginal product) changes in response to technology. I proved that under mild assumptions, changes in technology induced by small changes in factor supplies are always (absolutely) biased towards the factor that has become more abundant. I also showed that under supermodularity-type assumptions, the same result can be generalized to any magnitude of change in factor supplies, and can be applied to simultaneous changes in the supplies of a set of factors.

Finally, the last section illustrated the possibility of strong (absolute) equilibrium bias. In particular, with strong equilibrium bias, an increase in the supply of a factor induces a sufficiently large change in technology so that the marginal product (price) of the factor that has become more abundant increases; in other words, demand curves for factors become *upward sloping.* The analysis demonstrated that such strong equilibrium bias is impossible without non-convexities, but is easy to obtain once non-convexities are present. Moreover, Theorem 8 provided precise conditions for such strong bias to exist, related to the failure of joint concavity of the F function in factor demands and technology, which is possible (and in fact quite typical) in equilibrium environments.

To keep the exposition simple, I have made differentiability assumptions throughout the paper, but the global results can be easily generalized by relaxing differentiability since they were derived using tools from the theory of monotone comparative statics. Another possible generalization is to introduce multiple goods rather than a single final good. This complicates the analysis, but the general insights do not appear to depend on the single good assumption. Yet another interesting generalization might be to integrate some of these results into growth models where there can be long-run growth due to technological change (see Acemoglu, 1998, 2002, 2003b or Jones, 2005, for various growth models with relative equilibrium bias). More important directions for future research include an investigation of the bias of technology in alternative settings where the problem of determining equilibrium technology is not equivalent to a maximization problem. The most important example of this is a strategic setting where there is (oligopolistic) competition between various firms that are also choosing their technologies. Finally, the most important area for future research is an empirical investigation of whether the implications of these strong theorems actually hold in the data.

7 Appendix A: Technology Choice with Monopoly in Economy C

In this appendix, I briefly discuss the results in the environment of Economy C if the research is undertaken by a profit-maximizing monopolist. The main result is that if we allow the monopolist to charge price schedules rather than a linear price, the result is once again an equilibrium that corresponds to the maximization of some function $F(\bar{\mathbf{Z}}, \bar{\mathbf{L}}, \theta)$.

Recall that the production function is given by (6), with the same assumptions on the function G. Suppose that without buying the rights to use some technology, each firm would produce zero output. They can buy these rights from the monopolist technology producer, at some price χ (I will specify what this price is a function of below). The major difference is that the technology monopolist will choose the technology θ to maximize its profits rather than social surplus or total output. Since without the technology, a firm produces zero output, the technology monopolist can charge each firm up to a price of $\pi(\mathbf{Z}^i, \mathbf{L}^i, \theta^i)$. Therefore, its profits are

$$\Pi(\theta) = \int_{i \in \mathcal{F}} t(i)\, \pi(\mathbf{Z}^i, \mathbf{L}^i, \theta^i) di - C(\theta),$$

where $t(i)$ is an indicator for whether firm i is buying the new technology. Under the same assumption as in Economy C that the monopolist can only choose one technology from the menu, it will simply maximize $\Pi(\theta)$. The problem here is that as the monopolist provides better technologies to all firms, they compete more fiercely for the factors of production, so factor prices increase, and as a result, the profits that the technology monopolist can extract decline. For example, if G exhibits constant returns to scale in (Z, L), the monopolist can never extract any positive profits by charging any price schedule $\chi(\theta)$ and selling its new technology to all firms. In fact, in this case, it would clearly be beneficial for the monopolist to charge a price that is not only a function of the technology, but also of the employment levels of the firms, so as to manipulate their factor demand. In particular, suppose that the monopolist can charge firm i a price $\chi\left(\mathbf{Z}^i, \mathbf{L}^i, \theta\right)$, which is the fee to use technology θ conditional on firm i employing $\left(\mathbf{Z}^i, \mathbf{L}^i\right)$. Now consider the following price function for the monopolist

$$\chi\left(\mathbf{Z}^i, \mathbf{L}^i, \theta\right) = \begin{cases} \pi(\bar{\mathbf{Z}}, \bar{\mathbf{L}}, \theta^i) - \xi & \text{if } \left(\mathbf{Z}^i, \mathbf{L}^i\right) = \left(\bar{\mathbf{Z}}, \bar{\mathbf{L}}\right) \\ \infty & \text{if } \left(\mathbf{Z}^i, \mathbf{L}^i\right) \neq \left(\bar{\mathbf{Z}}, \bar{\mathbf{L}}\right) \end{cases}$$

for some $\xi > 0$, and the strategy of selling to a total of $1 - \varepsilon$ firms, where $\varepsilon > 0$. The price schedule makes it profitable for all firms that are offered the technology to take it, since they will make additional profits equal to ξ by doing so. Since $\varepsilon > 0$, there will be excess labor supply, so all factor prices will be equal to 0. Consequently,

$$\sup_{\xi,\varepsilon,\theta} \Pi(\theta) = G(\bar{\mathbf{Z}}, \bar{\mathbf{L}}, \theta) - C(\theta).$$

This is written as "sup" not as "max", since the supremum is never reached and the monopolist approaches it as $\xi \downarrow 0$ and $\varepsilon \downarrow 0$. The important result for the analysis is that technology choice is again the solution to the maximization of some function $F(\bar{\mathbf{Z}}, \bar{\mathbf{L}}, \theta)$ (though the supremum is never reached). Even though in this case factor prices are equal to zero, all the results in the text apply to the marginal products of the factors (which are never zero).

8 Appendix B: Some Technical Definitions and Results

In this section, I define some of the terms used in the analysis of global equilibrium bias. The reader is referred to the much more detail discussion in Topkis (1978, 1998), and also to Milgrom and Roberts (1990) and Milgrom and Shannon (1994). At the end of the section, I also prove a lemma on negative semi-definite matrices, which is used in the proofs of Theorems 6 and 8.

Let X be a partially ordered set, with an order (reflexive, anti-symmetric and transitive binary relation) denoted by $\geq$ (or $>$). For example, $X = \mathbb{R}^2$ with the order such that $(x_1', x_2') \geq (>) (x_1, x_2)$ only if $x_1' \geq (>) x_1$ and $x_2' \geq (>) x_2$ is a partially ordered set. In contrast, $X = \mathbb{R}$ with the natural order $\geq$ $(>)$ is an ordered set or a chain. Let $x' \vee x$ denote the *join*, or the least upper bound of two elements of a partially ordered set X. For example, when $X = \mathbb{R}^2$, $(x_1', x_2') \vee (x_1, x_2) = (\max\{x_1, x_1'\}, \max\{x_2, x_2'\})$. Similarly, the *meet*, or the greatest lower bound of two elements of a partially ordered set is denoted by $x' \wedge x$, and for the case where $X = \mathbb{R}^2$, $(x_1', x_2') \wedge (x_1, x_2) = (\min\{x_1, x_1'\}, \min\{x_2, x_2'\})$. X or a subset S of X is a *lattice* if it contains the join and the meet of each pair of its elements. A subset X' of X is a *sublattice* of X (i.e., a lattice according to the same order over X) if X' contains the joint and the meet of each pair of its own elements.

Let $f : X \to \mathbb{R}$ be a real-valued function and X be a lattice. Then we have a more general definition of supermodularity than the one in the text:

Definition 14 A real-valued function $f(x)$ defined on a (sub)lattice X is supermodular if

$$f(x') + f(x'') \leq f(x' \vee x'') + f(x' \wedge x'') \tag{45}$$

for all x',$x'' \in X$. Moreover, $f(x)$ is strictly supermodular if it satisfies (45) with strict inequality for all unordered x',$x'' \in X$.

When $f(x)$ is twice continuously differentiable over X, the definition for supermodularity is equivalent to the one in the text.

Another useful definition is that of increasing differences, which weakens the supermodularity requirements.[32]

Definition 15 Let X and T be partially ordered sets. Then a function $f(x,t)$ defined on a subset S of $X \times T$ has *increasing differences in* (x,t), if for all $t'' > t$, $f(x,t'') - f(x,t)$

[32] The notion of "increasing differences" was originally called isotone in Topkis (1968, 1978), and is sometimes referred to as non-decreasing differences (e.g., Amir, 1996).

is nondecreasing in x. Moreover, $f(x,t)$ has *strictly increasing differences in* (x,t), if for all $t'' > t$, $f(x,t'') - f(x,t)$ is strictly increasing in x.

Clearly, (strictly) increasing differences in in (x,t) and in (t,x) are identical.

In the text, I also made use of the concepts of single crossing property and quasi-supermodularity. These are defined as follows:

Definition 16 A real-valued function $f(x)$ defined on a (sub)lattice X is quasi-supermodular if for all x',$x'' \in X$,

$$\begin{aligned} f(x') &\leq f(x' \vee x'') \Longrightarrow f(x'') \leq f(x' \wedge x''), \text{ and} \\ f(x') &< f(x' \vee x'') \Longrightarrow f(x'') < f(x' \wedge x''). \end{aligned} \tag{46}$$

Definition 17 Let $f(x,t)$ be a real-valued function defined on $X \times T$ where X and T are partially ordered sets. Then $f(x,t)$ satisfies the single crossing property in (x,t) if $x'' > x'$, $t'' > t'$ and $f(x'',t') \geq f(x',t')$ implies that $f(x'',t'') \geq f(x',t'')$ and $f(x'',t') > f(x',t')$ implies that $f(x'',t'') > f(x',t'')$.

We have the following result linking supermodularity to increasing differences.

Lemma 2 Suppose that X is a lattice. If $f(x)$ is (strictly) supermodular on X, then $f(x)$ exhibits (strictly) increasing differences on X. Moreover, suppose in addition that $X \subset \mathbb{R}^K$. Then if $f(x)$ exhibits (strictly) increasing differences on X, $f(x)$ is (strictly) supermodular on X.

Proof. The first part follows from Theorem 2.6.1 of Topkis (1998), while the second part is an implication of Corollary 2.6.1 of Topkis (1998). ■

Next, it is useful to state some of the relationships between these concepts invoked in Example 3, and some additional results linking increasing differences to the single crossing property:

Lemma 3 Let f be a real valued function. Then:

1. If X is a lattice and $f(x)$ is supermodular on X, then $f(x)$ is quasi-supermodular on X.

2. If X_1 and X_2 are lattices and X is a sublattice of $X_1 \times X_2$ and $f(x)$ is quasi-supermodular on X, then $f(x)$ has the single crossing property in (x_1,x_2) and (x_2,x_1) on X.

3. If X_1 and X_2 are chains and X is a sublattice of $X_1 \times X_2$ and $f(x)$ has the single crossing property in (x_1, x_2) and (x_2, x_1) on X, then $f(x)$ is quasi-supermodular on X.

4. If X_1 and X_2 are partially ordered sets, X is a subset of $X_1 \times X_2$ and $f(x_1, x_2)$ exhibits increasing differences in (x_1, x_2) on X, then $f(x_1, x_2)$ has the single crossing property in (x_1, x_2) and (x_2, x_1) on X.

Proof. See Lemma 2.6.5 of Topkis (1998). ■

The key theorem for the analysis is the Monotonicity Theorem of Topkis. Here I state a version, which combines elements from Topkis' (1998) Theorems 2.7.1, 2.8.1, 2.8.4, 2.8.6 and Corollary 2.7.1 (see also Topkis, 1978, Theorems 6.1, 6.2 and 6.3). Instead of striving for ultimate generality, I state a version that applies in the context of the problem in the text (e.g., instead of upper semi-continuity, which is necessary for the existence of solutions, I impose continuity etc.).

Theorem 10 *(Monotonicity Theorem)* Suppose that X and T are lattices and $f(x, t)$ is quasi-supermodular in x and exhibits increasing differences in (x, t) on a compact and complete sublattice S of $X \times T$ and continuous in x on S, then $A(t) \equiv \arg\max_{x \in S} f(x, t)$ is a non-empty, compact and complete sublattice of X and is increasing in t. Moreover, if $f(x, t)$ is quasi-supermodular in x and exhibits strictly increasing differences in (x, t), then for any $t' > t$, $x(t) \in A(t)$ and $x(t') \in A(t')$, $x(t') \geq x(t)$.

Proof. See Topkis (1998). ■

In the text, I make use of the second part of this theorem which requires $f(x, t)$ to exhibit strictly increasing differences in (x, t), which is only a slightly stronger requirement than increasing differences. With increasing differences, all the results in the text continue to apply except that we only know that the set $A(t)$ is increasing (ascending) in t, so all the comparisons have to be for the greatest or the least element of the set $A(t)$. Strict increasing differences ensures that any element of $A(t')$ is greater than any element of $A(t)$ for $t' > t$.

Another useful theorem, first derived by Amir (1996) and generalized by Topkis (1998) Theorem 2.8.5, is the following, which I refer to as the "Strong Monotonicity Theorem". Here again I state a slightly simplified version of the theorem:

Theorem 11 *(Strong Monotonicity Theorem)* Suppose that X is a convex sublattice of $\mathbb{R}^n$ and T is a sublattice of $\mathbb{R}^m$, and $f(x, t)$ is quasi-supermodular in x, twice continuously differentiable in (x, t) on a compact and complete sublattice S of $X \times T$, continuous in x on S, and $\partial f(x,t)/\partial x_i$ is strictly increasing in t for all $i = 1,...,n$, then $A(t) \equiv \arg\max_{x\in S} f(x,t)$ is a non-empty, compact and complete sublattice of X. Moreover, if $t', t \in T$ with $t' > t$, and $x(t) \in A(t)$ and $x(t') \in A(t')$ are in the interior of X, then $x(t') > x(t)$.

Proof. See Topkis (1998) Theorem 2.8.5. ∎

The important feature of this strong monotonicity theorem is that under some additional assumptions, it establishes a strict ordering between $x(t')$ and $x(t)$, while the original monotonicity theorem only has a weak ordering.[33]

Finally, we have the following lemma, which is used in the proofs of Theorems 6 and 8.[34] Recall that for a matrix (vector) v, v' denotes its transpose.

Lemma 4 Consider the $n \times n$ matrix

$$B = \begin{pmatrix} C & v \\ v' & b \end{pmatrix}, \tag{47}$$

where C is a $(n-1) \times (n-1)$ symmetric negative definite matrix, b is a scalar, and v is a $(n-1) \times 1$ column vector. Then we have that B is negative semi-definite if and only if $b - v'C^{-1}v \leq 0$.

Proof. ($\Longleftarrow$) B is negative semi-definite if and only if

$$(x;y)'B(x;y) \leq 0,$$

where x is an arbitrary $(n-1) \times 1$ vector and y is a scalar, $(x;y)$ is the $n \times 1$ column vector constructed by stacking x and y. Using the form of B in (47), we have

$$(x;y)'B(x;y) = x'Cx + 2yx'v + by^2. \tag{48}$$

When $y = 0$, the above expression is always nonpositive since C is negative definite, so B is negative semi-definite as claimed.

Next consider the case where $y \neq 0$. In this case, let z be the $(n-1) \times 1$ vector constructed as $z = x/y$, and let us further expand (48):

$$\begin{aligned} (x;y)'B(x;y) &= y^2(z'Cz + 2z'v + b) \\ &= y^2(z'Cz + 2z'v + v'C^{-1}v) + y^2(b - v'C^{-1}v). \end{aligned} \tag{49}$$

[33] The assumption that $\partial f(x,t)/\partial x_i$ is strictly increasing in t for all $i = 1,...,n$, is slightly weaker than $\partial^2 f(x,t)/\partial x_i \partial t_j > 0$ for all $i = 1,...,n$ and $j = 1,...,m$. Also, the condition that $x(t')$ and $x(t)$ are in the interior of X can be relaxed along the lines of Theorem 2.8.5 of Topkis (1998).

[34] I thank Alp Simsek for help with the proof of this lemma.

Since C is a real symmetric negative definite matrix, $-C$ is a real symmetric and positive definite matrix, so there exists a non-singular matrix M such that $-C = M'M$. Moreover, we also have that $-C^{-1} = M^{-1}(M')^{-1} = M^{-1}(M^{-1})'$ [since $(M')^{-1} = (M^{-1})'$]. Now, rewriting equation (49) in terms of M, we have

$$\begin{aligned}(x;y)'B(x;y) &= -y^2(z'(-C)z - 2z'v - v'C^{-1}v) + y^2(b - v'C^{-1}v) \\ &= -y^2(z'(M'M)z - 2z'v + v'M^{-1}(M')^{-1}v) + y^2(b - v'C^{-1}v). \quad (50)\end{aligned}$$

(50) implies that B is negative semi-definite if and only if

$$\kappa \equiv y^2(z'(M'M)z - 2z'v + v'M^{-1}(M')^{-1}v) - y^2(b - v'C^{-1}v) \geq 0.$$

Now, rearranging terms and with straightforward matrix manipulation, we have

$$\begin{aligned}\kappa &\equiv y^2((Mz)'Mz - 2z'\left(M'(M')^{-1}\right)v + \left((M^{-1})'v\right)'(M^{-1})'v) - y^2(b - v'C^{-1}v), \\ &\equiv y^2((Mz)'Mz - 2(Mz)'(M^{-1})'v + \left((M^{-1})'v\right)'(M^{-1})'v) - y^2(b - v'C^{-1}v) \\ &\equiv y^2\left[\left(Mz - (M^{-1})'v\right)'\left(Mz - (M^{-1})'v\right)\right] - y^2(b - v'C^{-1}v).\end{aligned}$$

Therefore, B is negative semi-definite if and only if

$$\kappa \equiv y^2\left[\left(Mz - (M^{-1})'v\right)'\left(Mz - (M^{-1})'v\right)\right] - y^2(b - v'C^{-1}v) \geq 0. \quad (51)$$

Now suppose

$$b - v'C^{-1}v \leq 0,$$

then, from equation (51), the first term of κ takes the form $y^2a'a$ for $a \equiv \left(Mz - (M^{-1})'v\right)'\left(Mz - (M^{-1})'v\right)$ and is always non-negative for any z, so $\kappa \geq 0$, establishing that B is negative semi-definite.

($\Longrightarrow$) Conversely, suppose that B is negative semi-definite, which implies that $(x;y)'B(x;y) \leq 0$ for all $(x;y)$. To obtain a contradiction, suppose that

$$b - v'C^{-1}v > 0.$$

Then, take $y \neq 0$, and in terms of equation (51), set $z = M^{-1}(M')^{-1}v$, which yields $\kappa = -y^2(b - v'C^{-1}v) < 0$ in equation (51), contradicting the hypothesis that B is negative semi-definite (or that $(x;y)'B(x;y) \leq 0$ for all $(x;y)$), thus yielding a contradiction. ■

9 References

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